

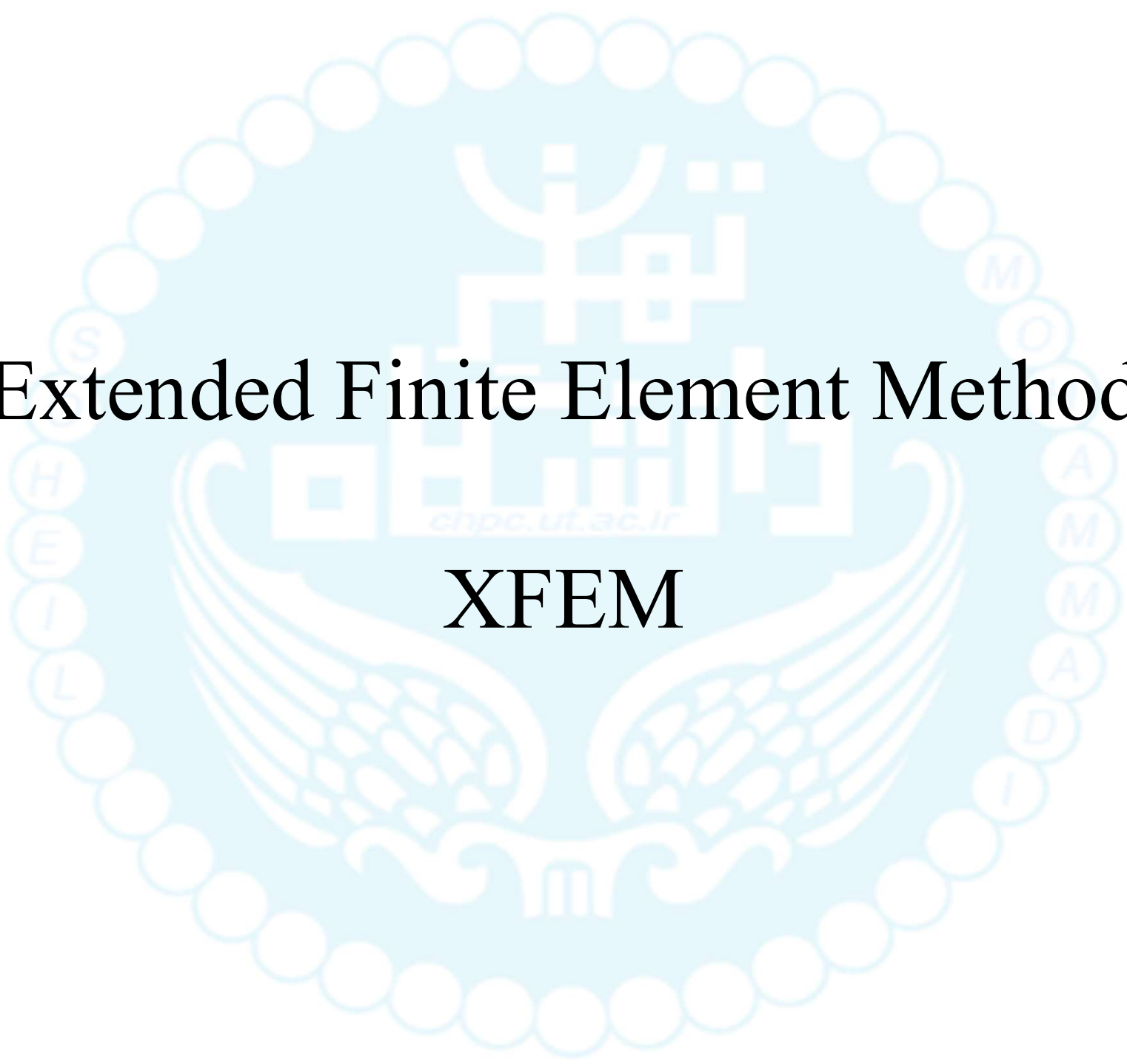


Fracture Mechanics

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Extended Finite Element Method

XFEM

XFEM FORMULATION

The basic concept of XFEM is to enrich the approximation space so that it becomes capable of reproducing certain features of the problem of interest, in particular discontinuities such as cracks or interfaces.

Although it is a local version of the partition of unity finite element enrichment applied only in a certain local subdomain, it has strongly relied on the development of extrinsic enrichments for crack simulations by a number of meshless methods such as EFG and Hp-clouds.

Naturally, the first XFEM approximations were also developed for simulation of strong discontinuities in fracture mechanics. This was later extended to include weak discontinuity and interface problems.

XFEM FORMULATION

XFEM can be assumed to be a classical FEM capable of handling arbitrary strong and weak discontinuities.

In the extended finite element method, first, the usual finite element mesh is produced. Then, by considering the location of discontinuities, a few degrees of freedom are added to the classical finite element model in selected nodes near to the discontinuities to provide a higher level of accuracy.

Basic XFEM approximation

Consider $\mathbf{x}$, a point in a finite element model. Also assume there is a discontinuity or singularity in the arbitrary domain discretised into some n node finite elements.

In the extended finite element method, the following approximation is utilised to calculate the displacement for the point $\mathbf{x}$ locating within the domain (Belytschko and Black 1999)

$$\mathbf{u}^h(\mathbf{x}) = \mathbf{u}^{\text{FE}} + \mathbf{u}^{\text{enr}} = \sum_{j=1}^n N_j(\mathbf{x}) \mathbf{u}_j + \sum_{k=1}^m N_k(\mathbf{x}) \psi(\mathbf{x}) \mathbf{a}_k$$

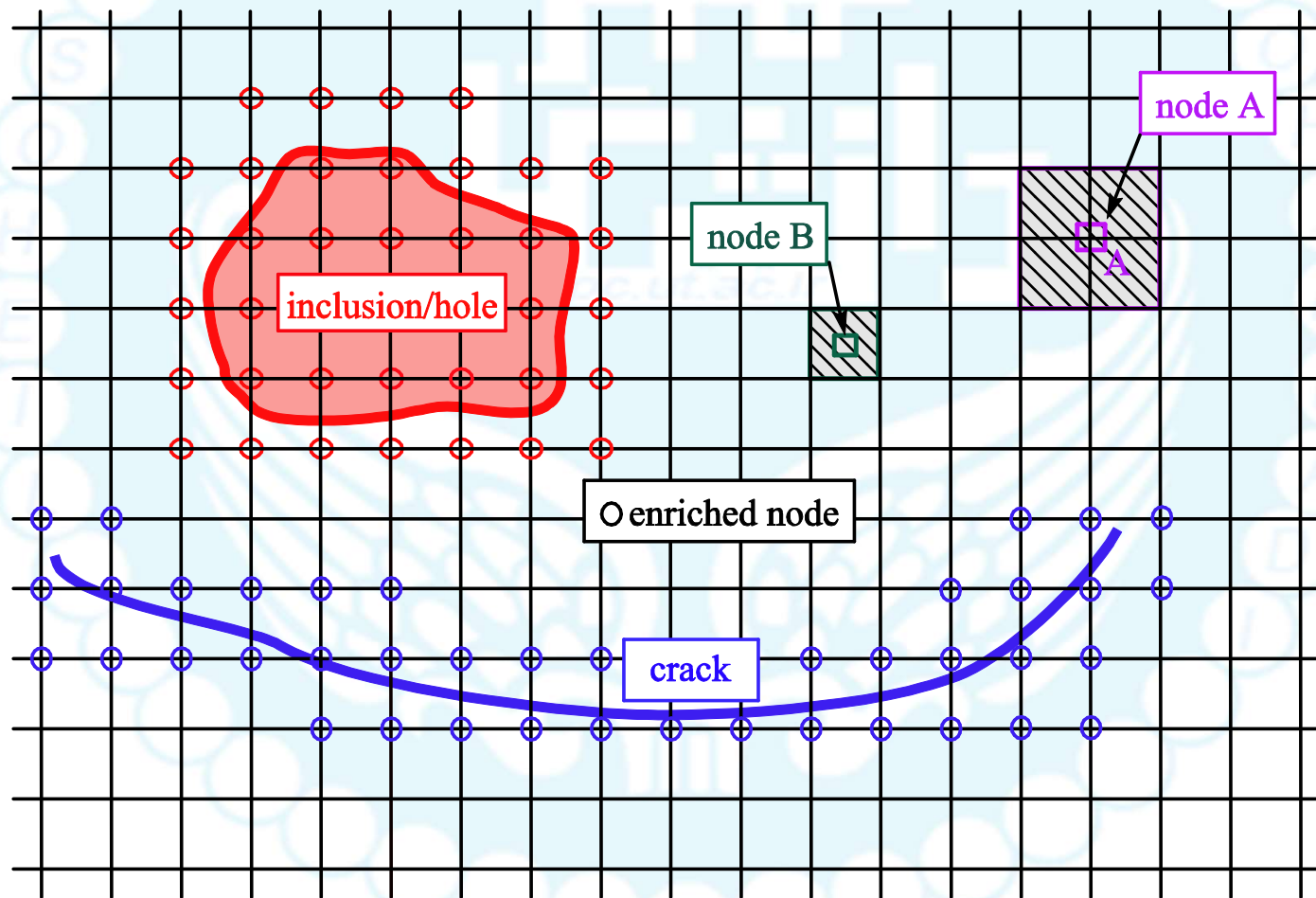
where $\mathbf{u}_j$ is the vector of regular degrees of nodal freedom in the finite element method, $\mathbf{a}_k$ is the set of degrees of freedom added to the standard finite element model and $\psi(\mathbf{x})$ is the set of enrichment functions defined for the set of nodes included in the influence (support) domain of the discontinuity or singularity.

$$\mathbf{u}^h(\mathbf{x}) = \mathbf{u}^{\text{FE}} + \mathbf{u}^{\text{enr}} = \sum_{j=1}^n N_j(\mathbf{x}) \mathbf{u}_j + \sum_{k=1}^m N_k(\mathbf{x}) \psi(\mathbf{x}) \mathbf{a}_k$$

While, in general, $m=n$ for standard XFEM approximation, there are cases where full enrichment of the element may not be possible to special circumstances. As a result, the PU condition may also be violated. This can well be accepted only locally and for a few elements, with more accurate solution than standard FEM solution.

It should be noted that the main goal for XFEM solution is to increase the accuracy of the solution. While for some problems, this increase leads to exact analytical solution, for many problems, only improved solution with higher accuracy is achieved, because no exact analytical solution is available.

For instance, the influence domain associated with node A consists of the elements containing that node, whereas for an interior node B (usually in higher order elements) it is the element surrounding the node.



$$\mathbf{u}^h(\mathbf{x}) = \mathbf{u}^{\text{FE}} + \mathbf{u}^{\text{enr}} = \sum_{j=1}^n N_j(\mathbf{x}) \mathbf{u}_j + \sum_{k=1}^m N_k(\mathbf{x}) \psi(\mathbf{x}) \mathbf{a}_k$$

The first term on the right-hand side is the classical finite element approximation to determine the displacement field, while the second term is the enrichment approximation which takes into account the existence of any discontinuities.

The second term utilizes additional degrees of freedom to facilitate modelling the existence of any discontinuous field, such as a crack, without modelling it explicitly in the finite element mesh.

$$\mathbf{u}^h(\mathbf{x}) = \mathbf{u}^{\text{FE}} + \mathbf{u}^{\text{enr}} = \sum_{j=1}^n N_j(\mathbf{x}) \mathbf{u}_j + \sum_{k=1}^m N_k(\mathbf{x}) \psi(\mathbf{x}) \mathbf{a}_k$$

The enrichment function $\psi(\mathbf{x})$ can be chosen by applying appropriate analytical solutions according to the type of discontinuity or singularity.

Some of the main objectives for using various types of enrichment functions within an XFEM procedure can be expressed as:

- Reproducing the singular field around a crack tip.
- Continuity in displacement between adjacent finite elements.
- Independent strain fields in two different sides of a crack surface or a material interface.
- Other features according to the specific discontinuity problem.

$$\mathbf{u}^h(\mathbf{x}) = \mathbf{u}^{\text{FE}} + \mathbf{u}^{\text{enr}} = \sum_{j=1}^n N_j(\mathbf{x}) \mathbf{u}_j + \sum_{k=1}^m N_k(\mathbf{x}) \psi(\mathbf{x}) \mathbf{a}_k$$

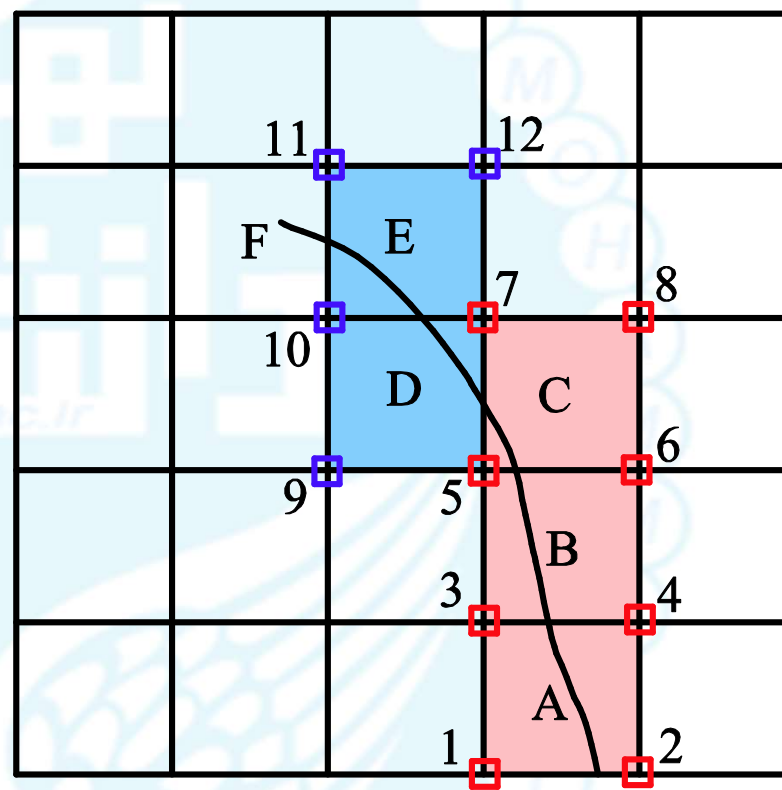
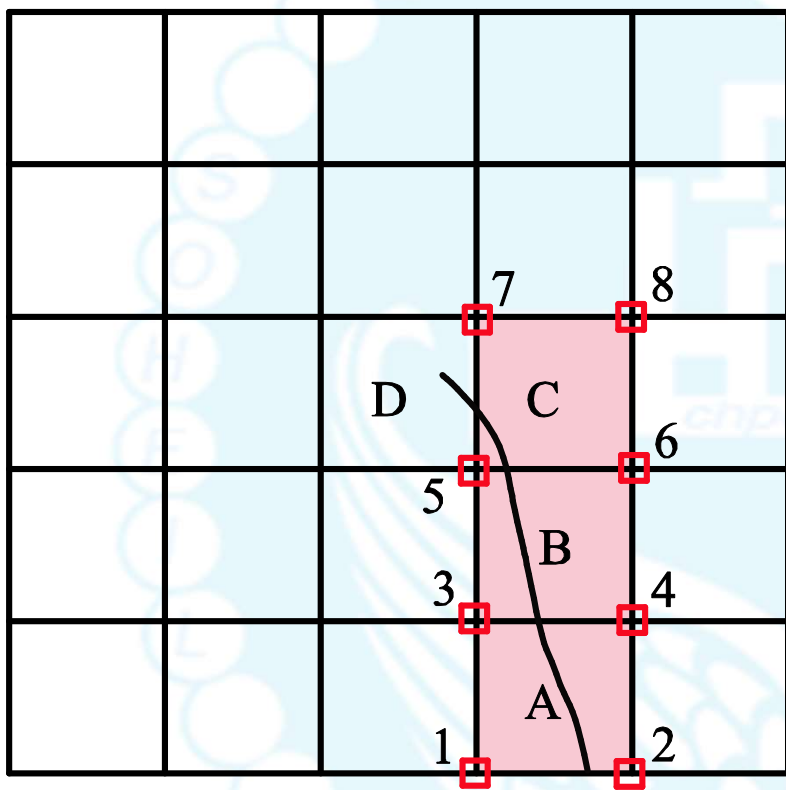
For np multiple discontinuities within a finite element, the XFM approximation can be further extended to:

$$\mathbf{u}^h(\mathbf{x}) = \mathbf{u}^{\text{FE}} + \mathbf{u}^{\text{enr}} = \sum_{j=1}^n N_j(\mathbf{x}) \mathbf{u}_j + \sum_{l=1}^{np} \sum_{k=1}^m N_k(\mathbf{x}) \psi^l(\mathbf{x}) \mathbf{a}_k^l$$

Selection of nodes for discontinuity enrichment

There have been different approaches for the selection of nodes to be enriched.

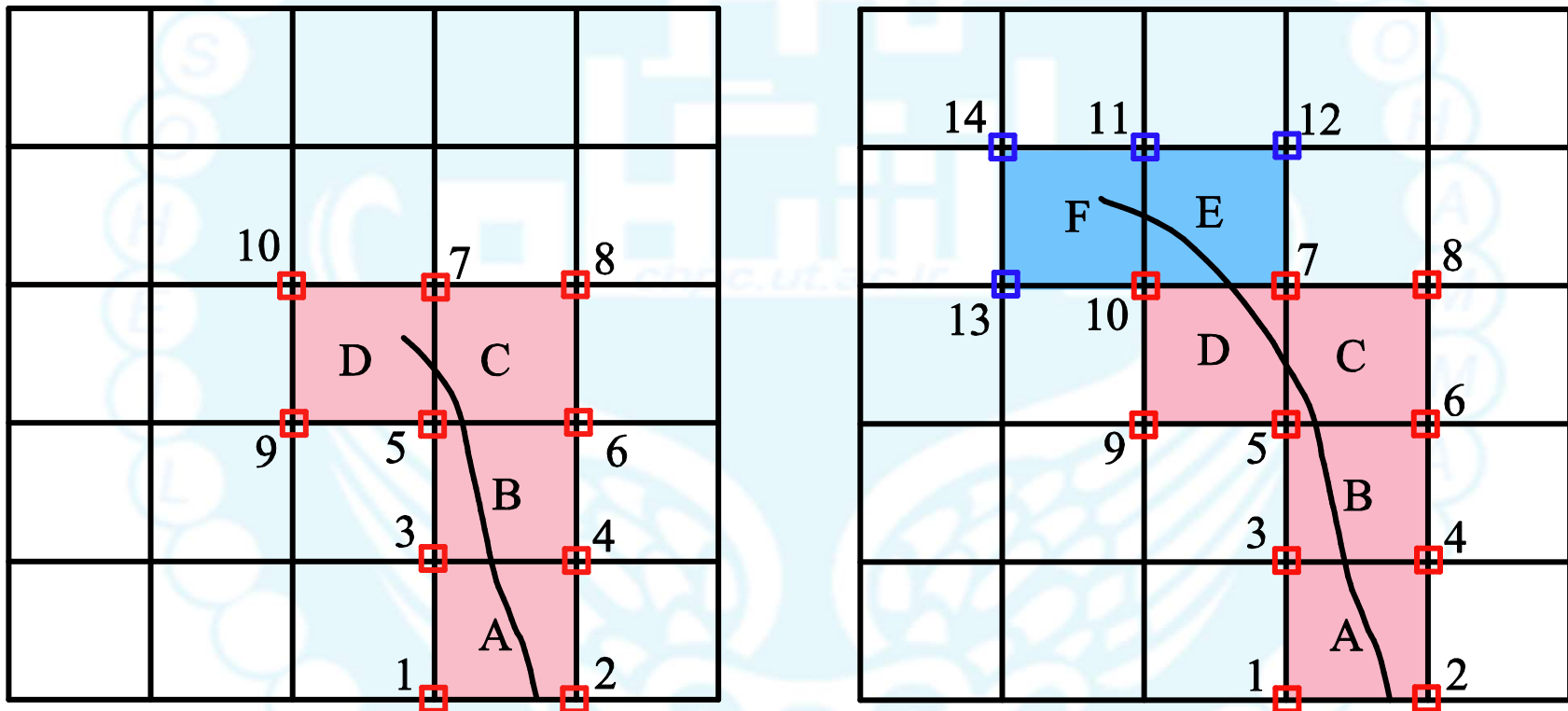
One method allows for the discontinuity to be modelled across the crack over the points along the crack surface. The value of the modified (enriched) shape function remains zero at all nodes and edges that do not intersect with the crack. This is important in satisfying the inter-element continuity requirements. This method only affects the element containing a crack, and does not directly influence other elements, even if they share a common node with the enriched element.



Heaviside enrichment type I

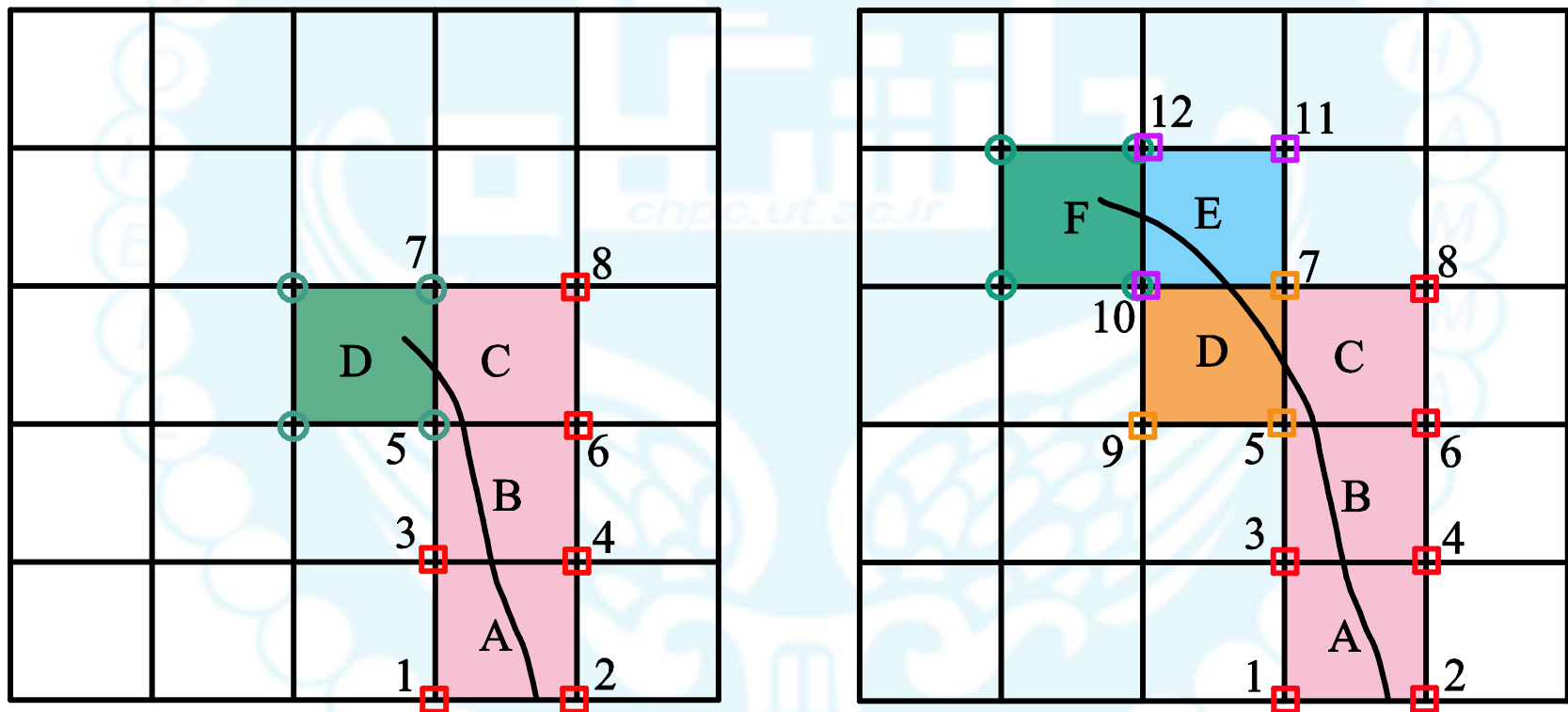
- A simple procedure for selection of nodes for enrichment is that at each stage of the propagation, nodes on edges cut by the crack path are enriched.
- Even if the crack tip locates just on an edge, the corresponding nodes are not enriched.
- A potential source of instability is when a crack path passes along the finite element edges.
- This technique adds two enrichment degrees of freedom to an element per any enriched node. As a result, for a quadrilateral element on the path of a crack, sixteen degrees of freedom (DOF) are assigned: eight classical DOFs and eight enriched DOFs.
- This approach is frequently used in complicated problems, such as cracking in shells, where an asymptotic analytical solution around the crack tip may not exist or difficult to be derived.

In an alternative procedure, the enrichment is applied onto all nodal points of elements that contain part of the crack.



Heavisde enrichment type II

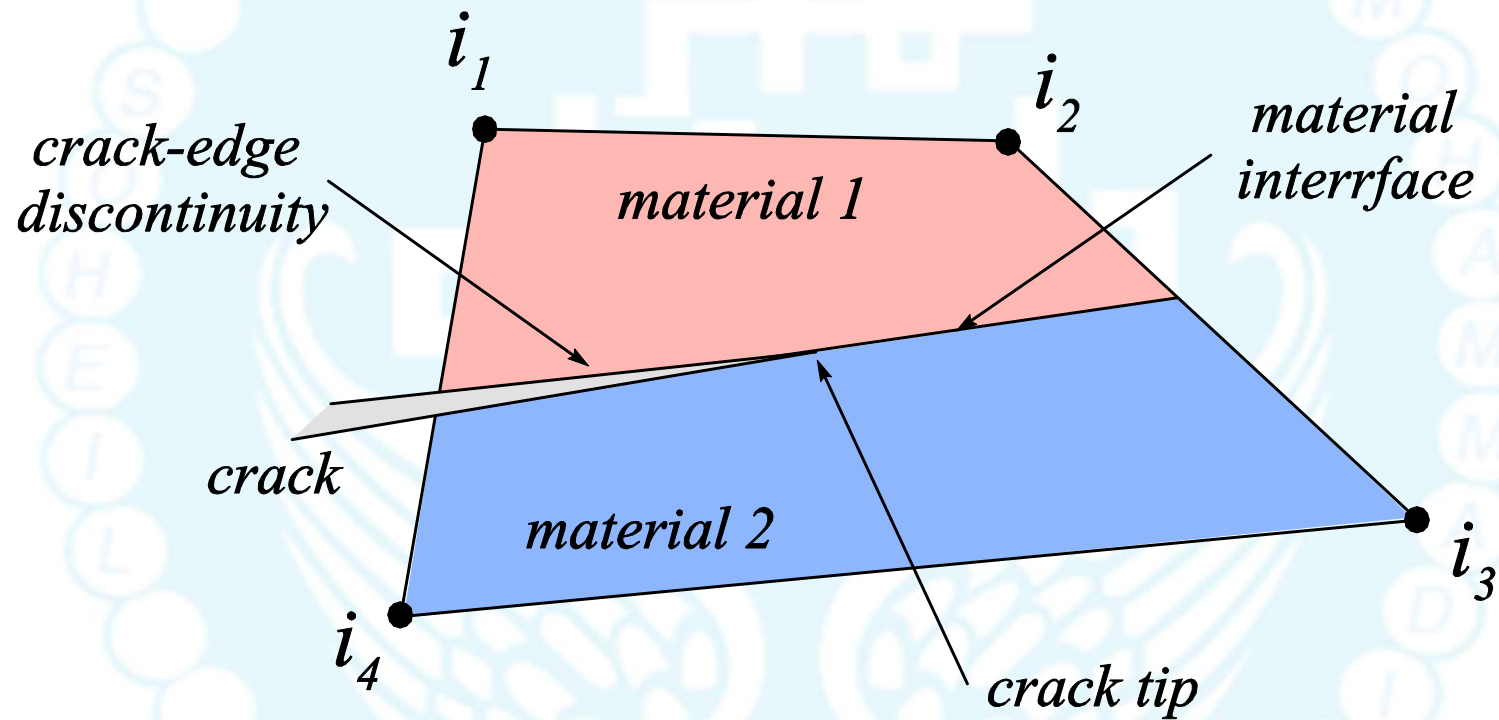
The most appropriate choice is to use both Heaviside and crack tip enrichment functions where required. The crack tip enrichment functions in the crack tip element also allow for accurate representation of discontinuity in that element. The crack does not affect other nodes and their associated classical finite element degrees of freedom.



Heaviside and crack tip enrichment functions

XFEM Modelling
of
a Cracked Bimaterial Element

XFEM modelling of the crack/material interface



XFEM modelling of the crack/material interface

In order to model crack surfaces and crack tips in the extended finite element method, the approximate displacement function $\mathbf{u}^h$ can be expressed in terms of the classical $\mathbf{u}$, crack-split $\mathbf{u}^H$, crack tip $\mathbf{u}^{tip}$ and material interface $\mathbf{u}^{mat}$ components as

$$\mathbf{u}^h(\mathbf{x}) = \mathbf{u}(\mathbf{x}) + \mathbf{u}^H(\mathbf{x}) + \mathbf{u}^{tip}(\mathbf{x}) + \mathbf{u}^{mat}(\mathbf{x})$$

or more explicitly

$$\begin{aligned} \mathbf{u}^h(\mathbf{x}) = & \left[\sum_{j=1}^n N_j(\mathbf{x}) \mathbf{u}_j \right] + \left[\sum_{h=1}^{mh} N_h(\mathbf{x}) H(x) \mathbf{a}_h \right] \\ & + \left[\sum_{k=1}^{mt} N_k(\mathbf{x}) \left(\sum_{l=1}^{mf} F_l(x) \mathbf{b}_k^l \right) \right] + \left[\sum_{m=1}^{mm} N_m(\mathbf{x}) \chi_m(x) \mathbf{c}_m \right] \end{aligned}$$

and in a simple shifted form for H and F functions to guarantee the interpolation:

$$\begin{aligned} \mathbf{u}^h(\mathbf{x}) = & \left[\sum_{j=1}^n N_j(\mathbf{x}) \mathbf{u}_j \right] \\ & + \left[\sum_{h=1}^{mh} N_h(\mathbf{x}) (H(x) - H(x_h)) \mathbf{a}_h \right] \\ & + \left[\sum_{k=1}^{mt} N_k(\mathbf{x}) \left(\sum_{l=1}^{mf} [F_l(x) - F_l(x_k)] \mathbf{b}_k^l \right) \right] \\ & + \left[\sum_{m=1}^{mm} N_m(\mathbf{x}) \chi_m(x) \mathbf{c}_m \right] \end{aligned}$$

n is the number of nodes of each finite element

mh is the number of nodes that have crack face (but not crack tip) in their support domain,

mt is the number of nodes associated with the crack tip in its influence domain,

mm is the number of nodes that are directly affected by the weak discontinuity.

$\mathbf{u}_j$ classical FEM DOFs

$\mathbf{a}_h$ is the vector of additional degrees of nodal freedom for modelling crack faces (not crack tips)

$\mathbf{b}_k^l$ is the vector of additional degrees of nodal freedom for modelling

$\mathbf{c}_m$ is the vector of additional degrees of nodal freedom for modelling weak discontinuity interfaces crack tips.

$N_j(\mathbf{x})$ are the FEM shape functions

$H(\mathbf{x})$ is the Heaviside function,

$F_l(\mathbf{x})$ are crack tip enrichment functions,

$\chi_m(\mathbf{x})$ is the enrichment function used for modelling weak discontinuities.



XFEM Enrichments

for

Strong Discontinuity

XFEM STRONG DISCONTINUITY ENRICHMENTS

In the extended finite element method, approximation of a discontinuous displacement field is based on the definition of specially designed shape functions by the use of enrichment functions.

The method operates on additional independent virtual degrees of freedom for the definition of the crack geometry and approximation of the displacement field.

It will then be combined with the classical finite element method to approximate the overall solution.

There have been a number of possible choices for the enrichment function $\psi(\mathbf{x})$. The following sections explain the basic ideas and discuss the various effects of enrichment on this simple one-dimensional problem.

Signed distance function $\xi(\mathbf{x})$

The most conventional form of the Heaviside function H is defined as,

$$H(\xi) = \text{sign}(\xi) = \begin{cases} 1 & \forall \xi > 0 \\ -1 & \forall \xi < 0 \end{cases}$$

where the signed distance function $\xi(\mathbf{x})$ can then be defined from the distance d of a point $\mathbf{x}$ to an interface Γ , as,

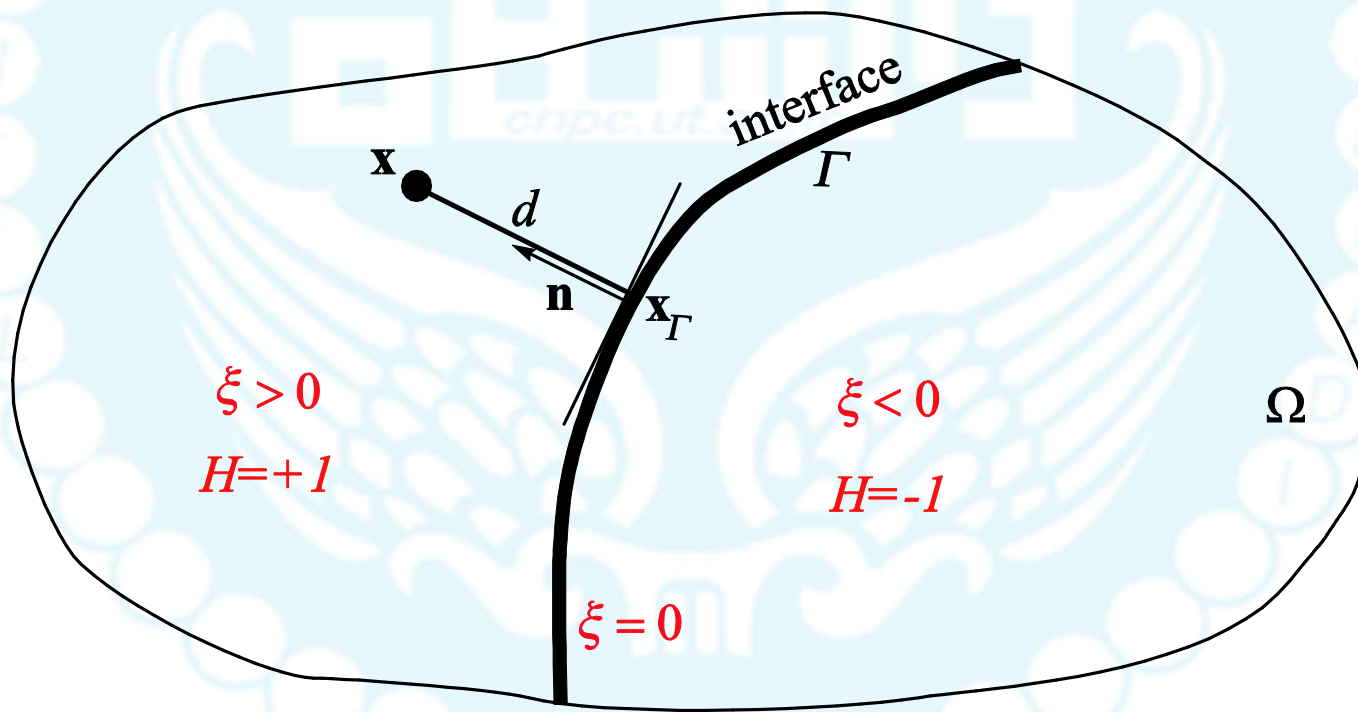
$$d = \|\mathbf{x} - \mathbf{x}_\Gamma\|$$

$$\xi(\mathbf{x}) = \underbrace{\min_{\mathbf{x}_\Gamma \in \Gamma} \|\mathbf{x} - \mathbf{x}_\Gamma\|}_{d} \text{sign}(\mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_\Gamma))$$

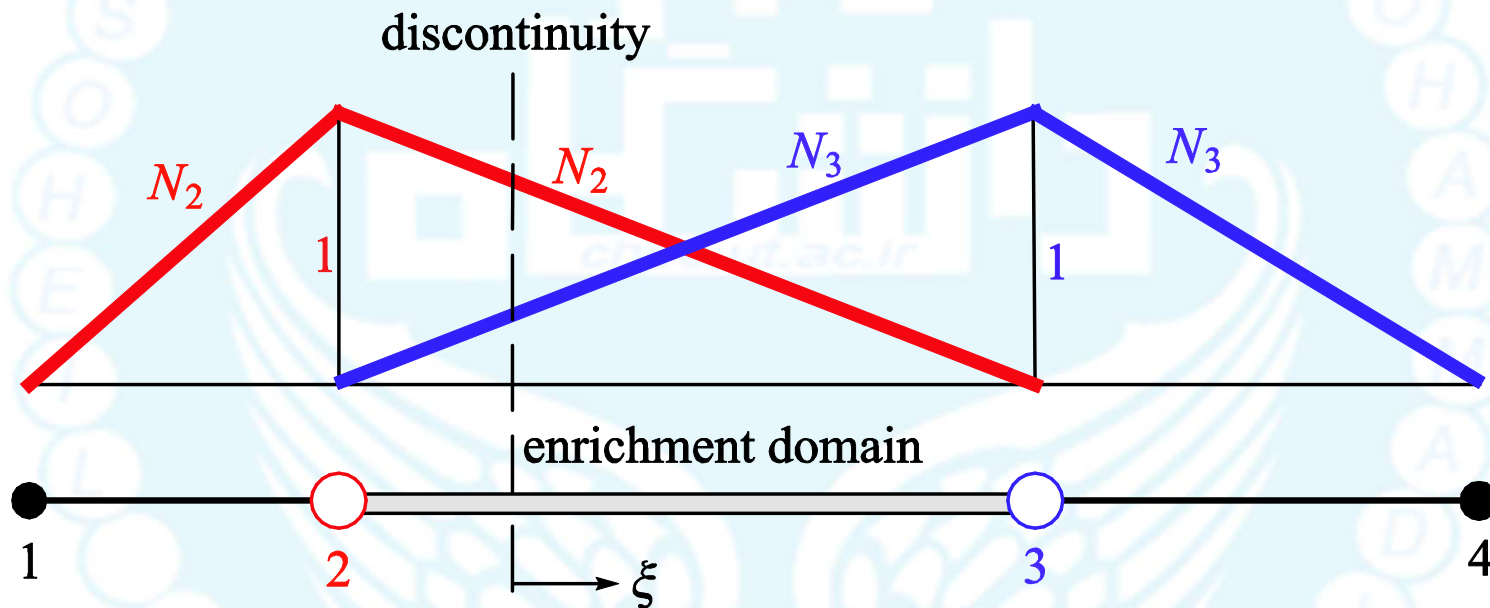
where $\mathbf{n}$ is the unit normal vector.

$$\xi(\mathbf{x}) = \underbrace{\min_{\mathbf{x}_\Gamma \in \Gamma} \|\mathbf{x} - \mathbf{x}_\Gamma\|}_{d} \operatorname{sign}(\mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_\Gamma))$$

$$H(\xi) = \operatorname{sign}(\xi) = \begin{cases} 1 & \forall \xi > 0 \\ -1 & \forall \xi < 0 \end{cases}$$



In order to discuss various effects of the modelling, a one-dimensional problem is considered which consists of four nodes and three linear 2-node finite elements with a strong discontinuity (crack) in an arbitrary location $x_c(\xi_c)$ within the middle element.



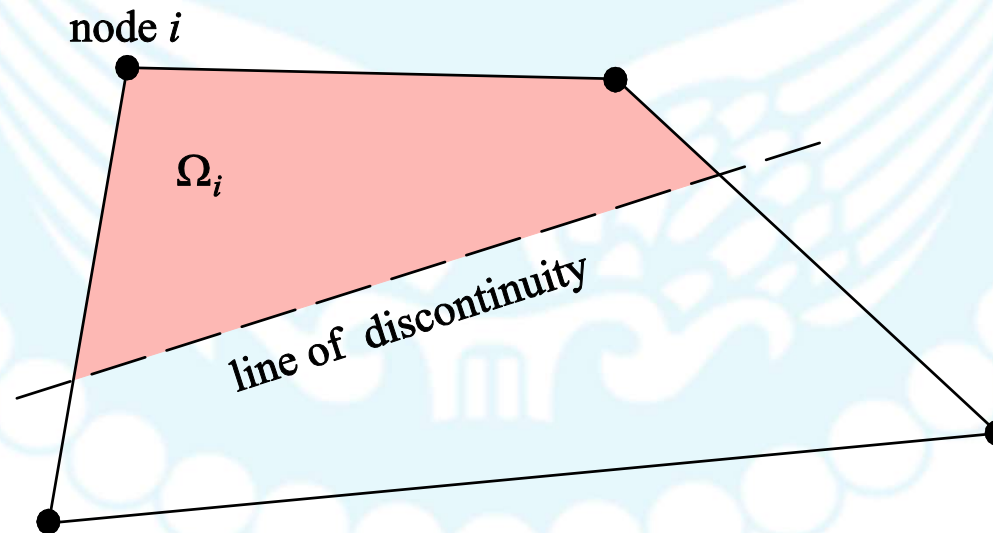
Only nodes 2 and 3 are required to be enriched, whereas nodes 1 and 4 are not influenced by the crack.

A modified FE shape function

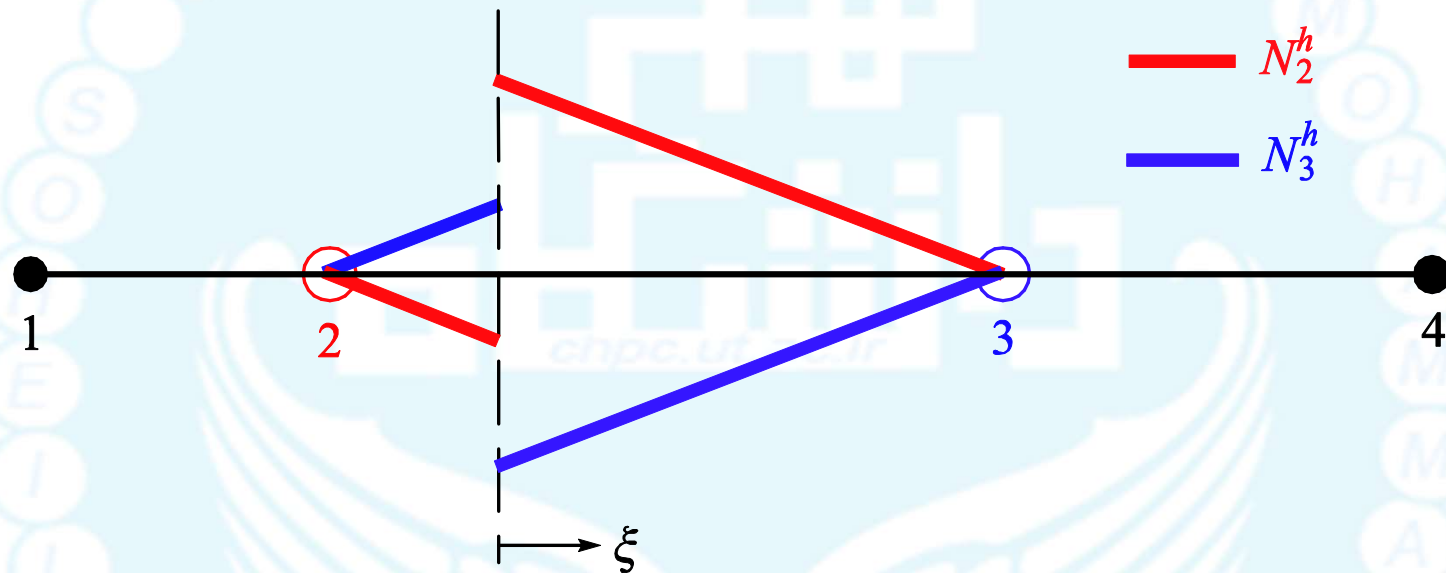
Earlier models used a simple modified shape function in the form of:

$$N_i^h = \begin{cases} N_i - 1 & \mathbf{x} \in \Omega_i \\ N_i & \mathbf{x} \notin \Omega_i \end{cases}$$

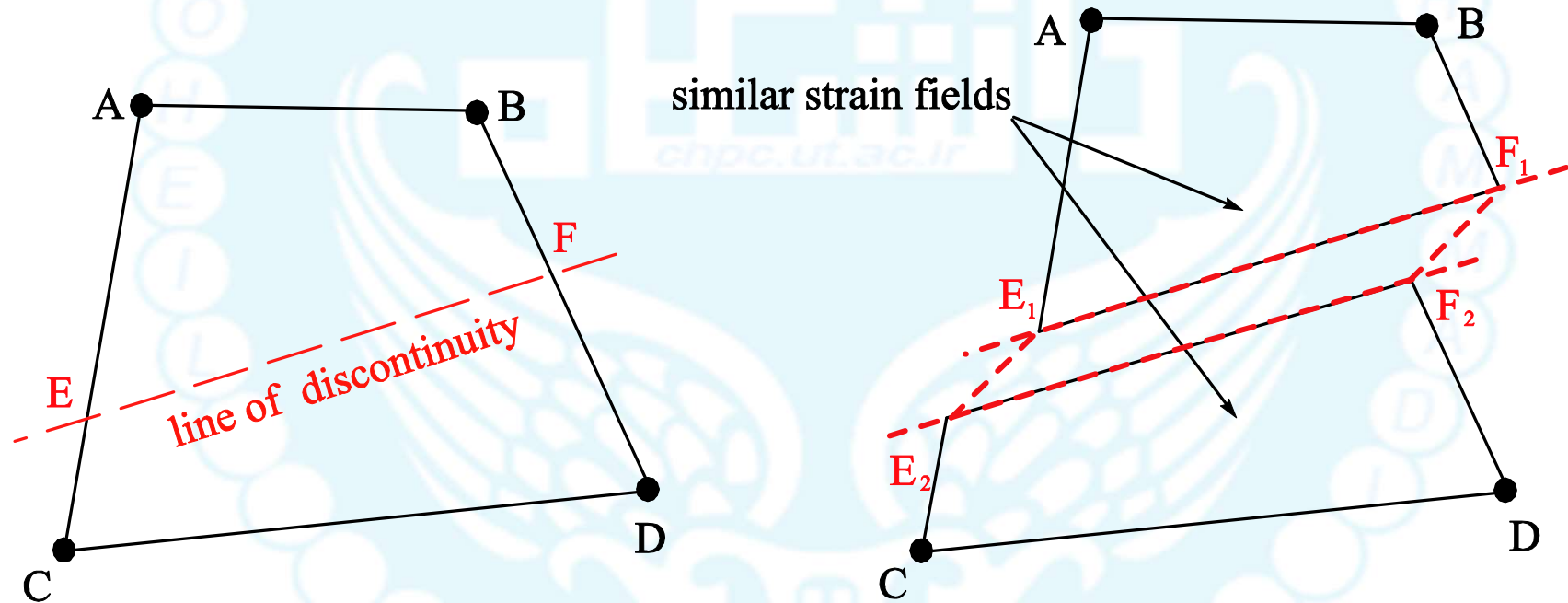
where N_i is the conventional finite element shape function and Ω_i is part of the element in between the crack and node i .



This jump enrichment can affect the shape functions for the simple one-dimensional case:

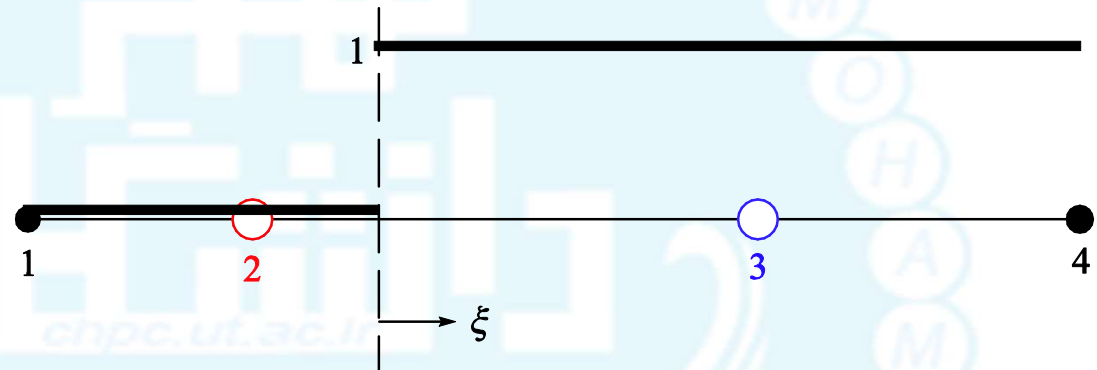


One problem with this type of jump function is that it provides similar strain fields (derivative of the displacement field) in both sides of the discontinuity. This is in contrast to the independent physical response of the segments, anticipated in a cracked element.

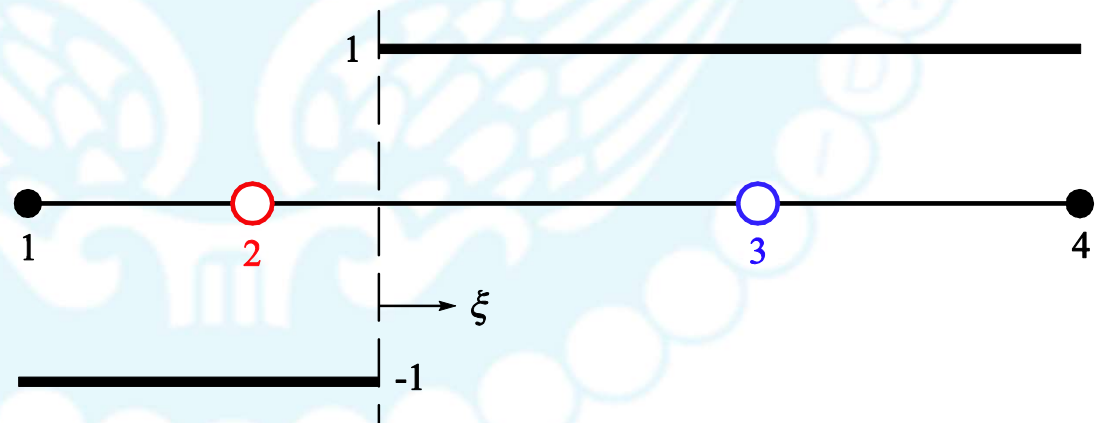


The Heaviside function

Different definitions have been adopted for the Heaviside function over the years.



a) step function

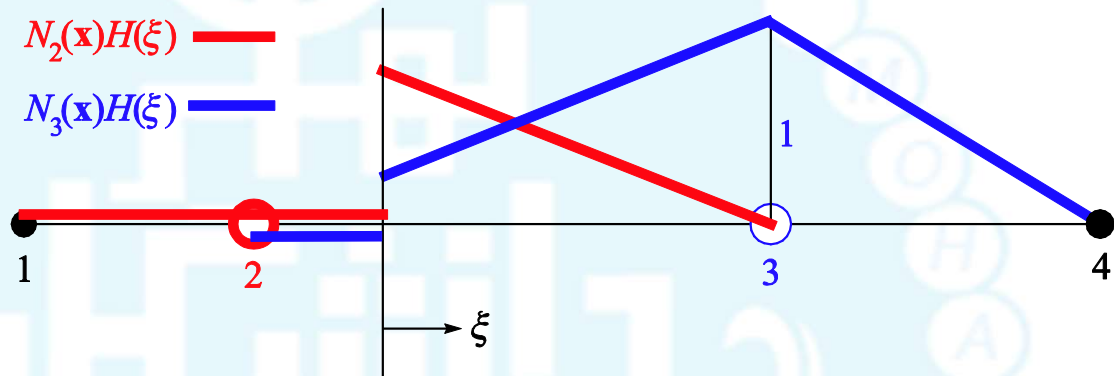


b) sign function

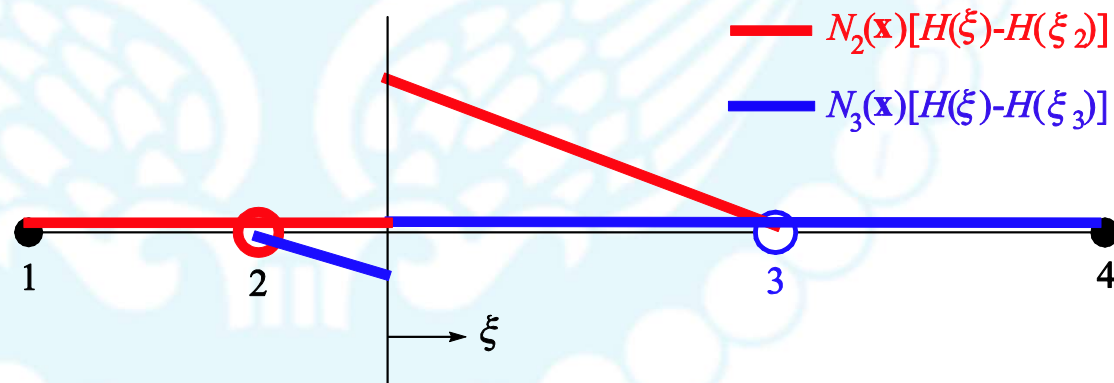
The step function

The first type of Heaviside function $H(\xi)$ is a step function,

$$H(\xi) = \begin{cases} 1 & \forall \xi > 0 \\ 0 & \forall \xi < 0 \end{cases}$$



a) effect of the step function on shape functions



b) effect of shifting on shape functions

XFEM approximation leads to

$$\mathbf{u}^h(\mathbf{x}) = \sum_{j=1}^n N_j(\mathbf{x}) \mathbf{u}_j + \sum_{k=1}^m N_k(\mathbf{x}) H(\xi) \mathbf{a}_k$$

To examine whether the approximation (3.77) is an interpolation, the value of the field variable $\mathbf{u}(\mathbf{x})$ on an enriched node i can be obtained as:

$$\mathbf{u}^h(\mathbf{x}_i) = \mathbf{u}_i + H(\xi_i) \mathbf{a}_i$$

which means that interpolation cannot be achieved and the nodal parameter $\mathbf{u}_i$ is not the real displacement value on the enriched node i .

A simple remedy to this shortcoming is to shift the step function around the node of interest:

$$\mathbf{u}^h(\mathbf{x}) = \sum_{j=1}^n N_j(\mathbf{x}) \mathbf{u}_j + \sum_{k=1}^m N_k(\mathbf{x}) (H(\xi) - H(\xi_k)) \mathbf{a}_k$$

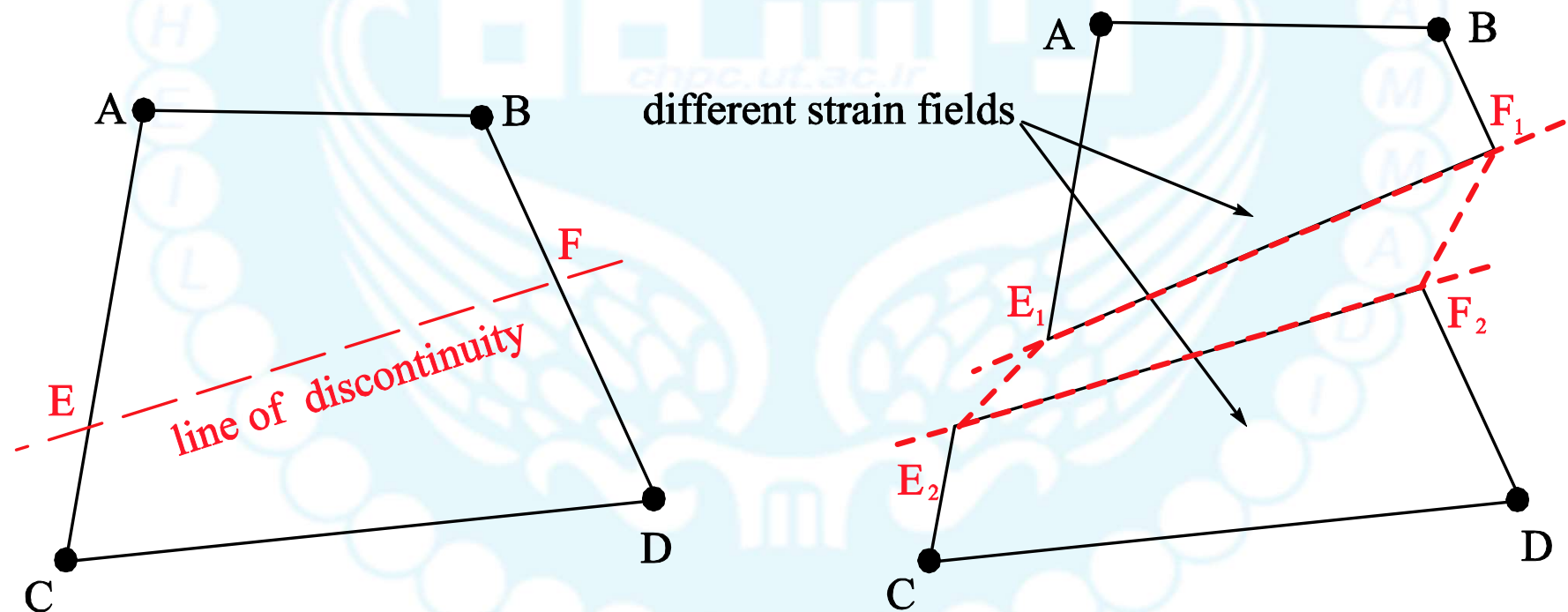
Consequently, interpolation can be automatically guaranteed.

The overall jump in the displacement field can be obtained from:

$$\langle \mathbf{u}^h(\mathbf{x}) \rangle = \mathbf{u}^h(\mathbf{x}^+) - \mathbf{u}^h(\mathbf{x}^-) = \cdots = \sum_{k=1}^m N_k(\mathbf{x}) \mathbf{a}_k$$

In the shifted case, the enriched shape functions vanish at both nodes 2 and 3. As a result, the direct effect of enrichment is limited only to the middle element which contains a discontinuity.

Application of the jump function on a quadrilateral element may lead to a discontinuous field, as depicted in Fig. 3.14. The deformation field includes all potential displacement fields independently for both sides of the crack. The strain fields also remain independent for both sides of the crack.



In order to avoid numerical instabilities, the following smoothed Heaviside functions can also be used for a small value of β less than the element size (Bordas and Legay, 2005):

$$H(\xi) = \begin{cases} 0 & \xi < -\beta \\ \frac{1}{2} + \frac{\xi}{2\beta} + \frac{1}{2\pi} \sin \frac{\pi\xi}{\beta} & -\beta < \xi < \beta \\ 1 & \xi > \beta \end{cases}$$

or

$$H(\xi) = \begin{cases} 0 & \xi < -\beta \\ \frac{1}{2} + \frac{1}{8} \left(9 \frac{\xi}{\beta} - 5 \frac{\xi^3}{\beta^3} \right) & -\beta < \xi < \beta \\ 1 & \xi > \beta \end{cases}$$

The derivative of the Heaviside function is the Dirac delta function $\delta(\xi)$:

$$H_{,i}(\xi) = \delta(\xi)$$

which can be approximated by the following smoothed functions

$$\delta(\xi) = \begin{cases} \frac{1}{2\beta} + \frac{1}{2\beta} \cos \frac{\pi\xi}{\beta} & -\beta < \xi < \beta \\ 0 & \text{otherwise} \end{cases}$$

or

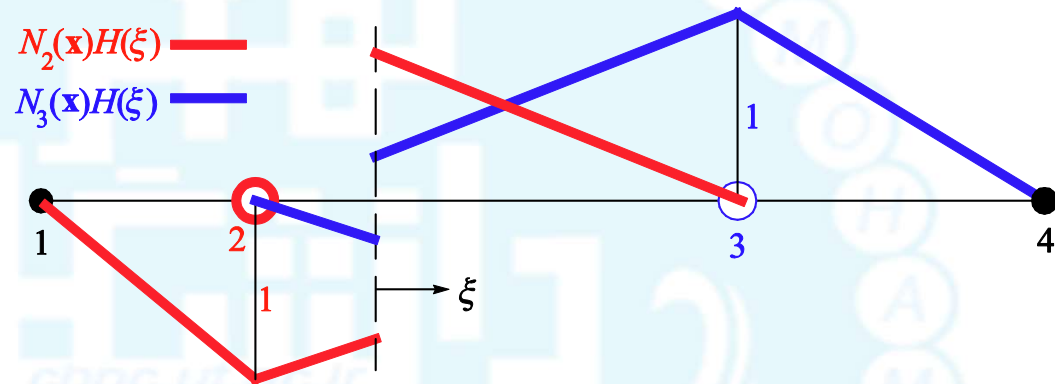
$$\delta(\xi) = \begin{cases} \frac{1}{8} \left(\frac{9}{\beta} - 15 \frac{\xi^2}{\beta^3} \right) & -\beta < \xi < \beta \\ 0 & \text{otherwise} \end{cases}$$

The sign function

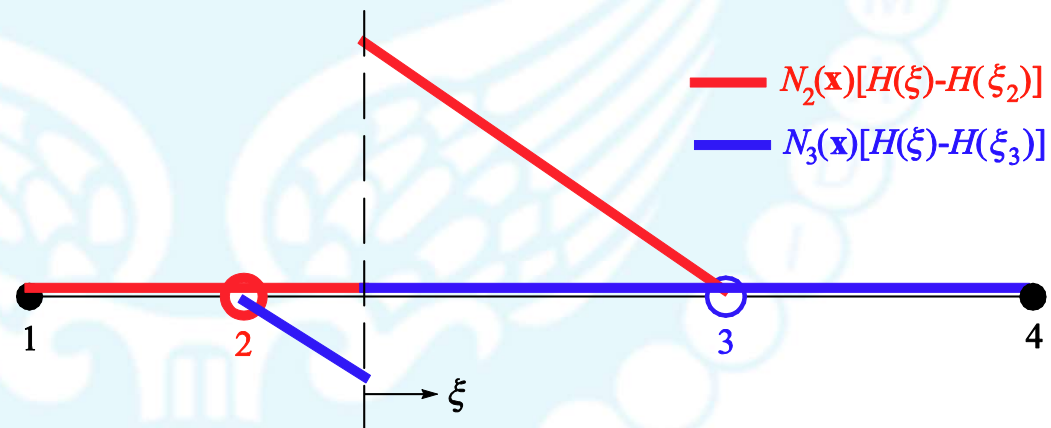
An alternative Heaviside enrichment function is the sign function

$$H(\xi) = \text{sign}(\xi)$$

$$= \begin{cases} 1 & \forall \xi > 0 \\ -1 & \forall \xi < 0 \end{cases}$$



a) effect of the sign function on shape functions



b) effect of shifting on shape functions

Again, this XFEM approximation is no longer an interpolation and the value of the field variable $u(\mathbf{x})$ on an enriched node i is not equal to the nodal value $\mathbf{u}_i$

$$\mathbf{u}^h(\mathbf{x}_i) = \mathbf{u}_i + H(\xi_i) \mathbf{a}_i \neq \mathbf{u}_i$$

A simple shifting procedure guarantees the interpolation:

$$\mathbf{u}^h(\mathbf{x}) = \sum_{j=1}^n N_j(\mathbf{x}) \mathbf{u}_j + \sum_{k=1}^m N_k(\mathbf{x}) (H(\xi) - H(\xi_k)) \mathbf{a}_k$$

It is observed that the enriched shape functions in the shifted form vanish at both nodes 2 and 3. As a result, the direct effect of enrichment is limited only to the middle element which contains a discontinuity.

The overall jump in the displacement field can be obtained from:

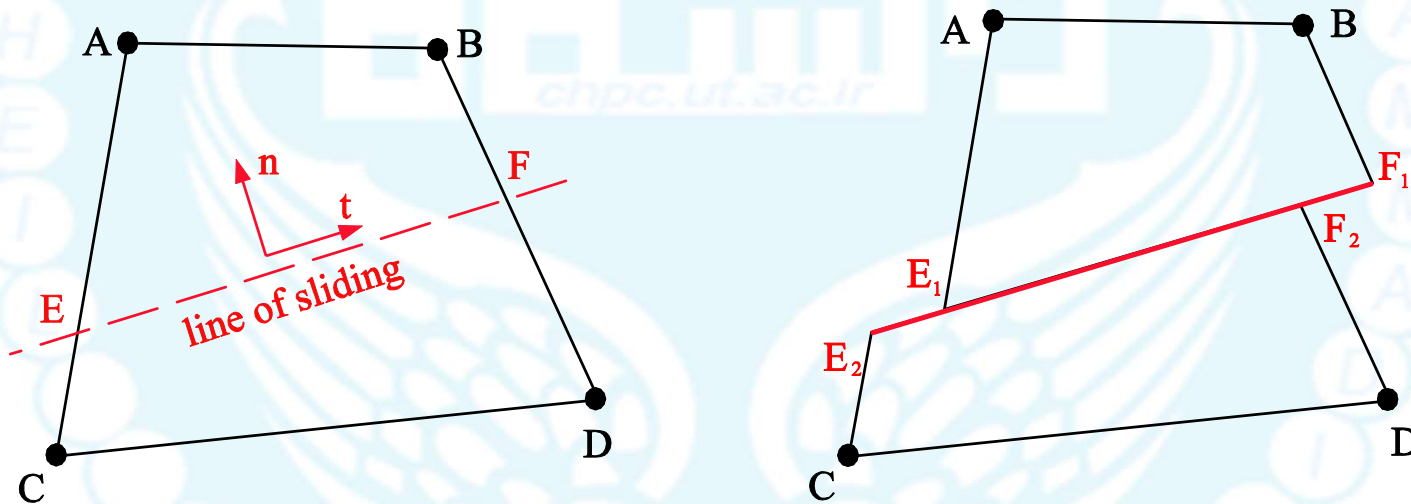
$$\langle \mathbf{u}^h(\mathbf{x}) \rangle = \mathbf{u}^h(\mathbf{x}^+) - \mathbf{u}^h(\mathbf{x}^-) = \dots = 2 \sum_{k=1}^m N_k(\mathbf{x}) \mathbf{a}_k$$

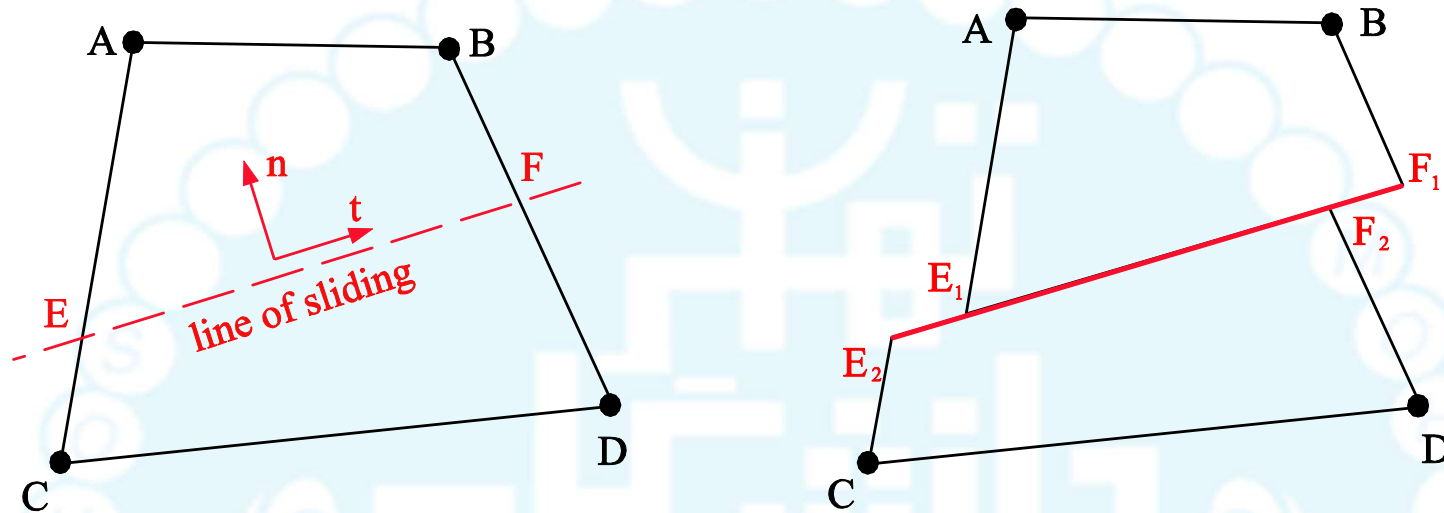
It is important to note that the choice of the jump in the enrichment function (sign or step functions) does not affect the overall solution.

Similar continuous forms can be derived for smoothed equivalent forms of the step function.

Strong tangential discontinuity

In a strong tangential discontinuity, a jump exists in the tangential direction whereas the displacement remains continuous in the normal direction.





The conventional approach for modelling a strong discontinuity by the Heaviside function is now modified to simulate the tangential discontinuity.

This is achieved by defining only one extra enrichment degree of freedom (per node) in the tangential direction (instead of standard two DOFs) (Daneshyar and Mohammadi, 2012).

The XFEM approximation can then be written as,

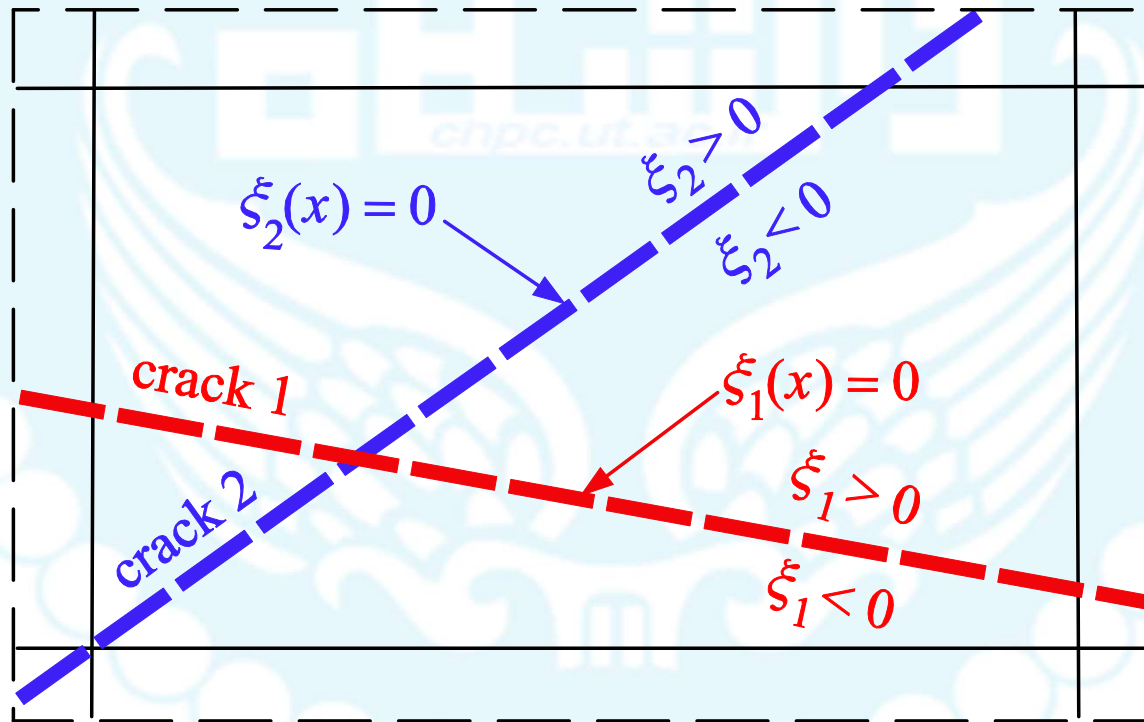
$$\mathbf{u}^h(\mathbf{x}) = \sum_{j=1}^n N_j(\mathbf{x}) \mathbf{u}_j + \sum_{h=1}^m N_h(\mathbf{x}) (H(\xi) - H(\xi_h)) \mathbf{a}_h \mathbf{t}_h$$

where $\mathbf{t}_h$ is the tangential direction unit vector.

All nodes of the finite elements that include a tangential discontinuity should be enriched by the tangential Heaviside enrichment functions.

Crack intersection

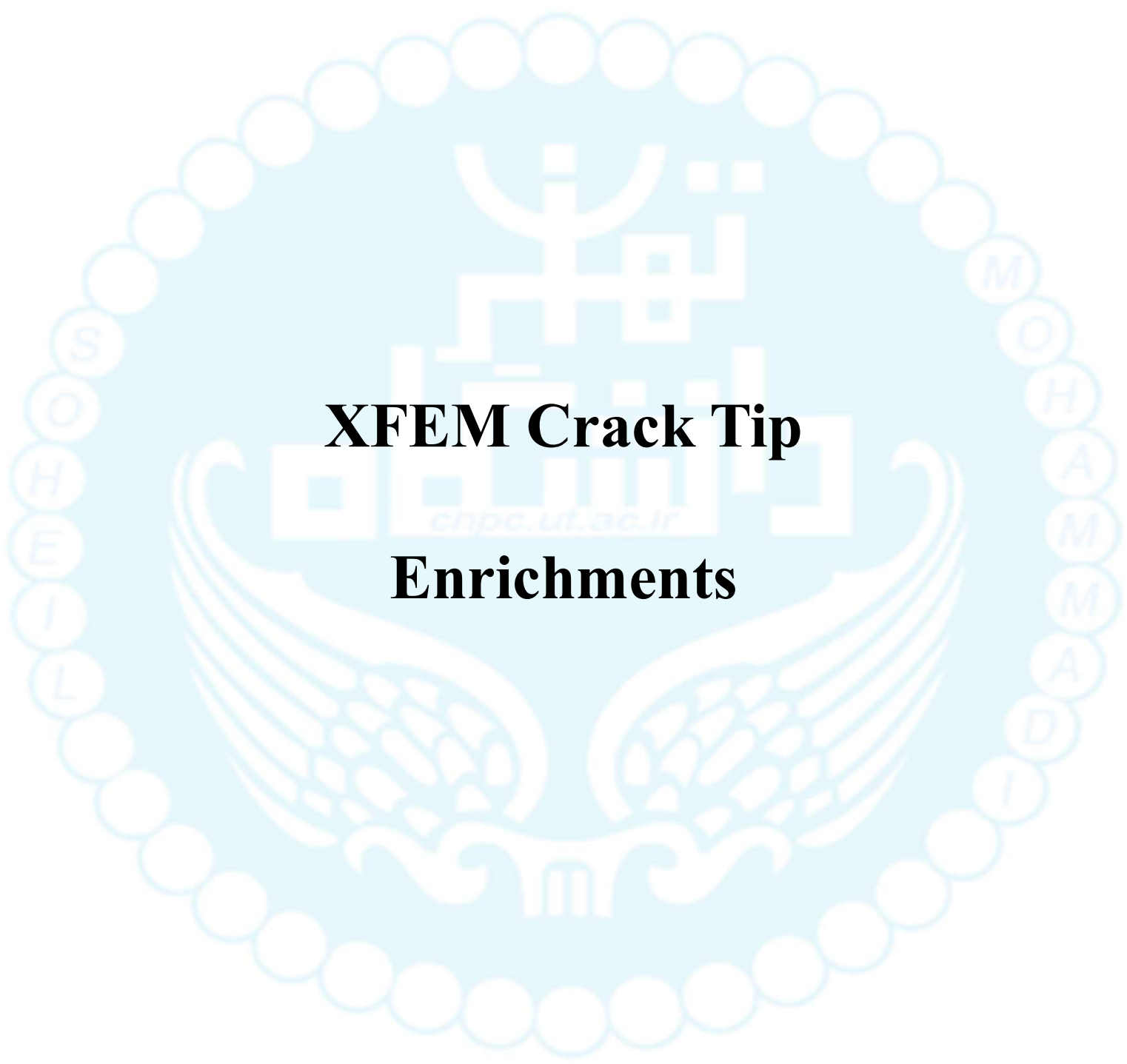
The basic equation for XFEM enrichment requires further modification if two or more cracks intersect within a finite element,



The XFEM approximation of the displacement for an element cut by two intersecting cracks may be written as (Daux *et al.* 2000):

$$\begin{aligned}\mathbf{u}^h(\mathbf{x}) = & \sum_{j=1}^n N_j(\mathbf{x}) \mathbf{u}_j \\ & + \sum_{h=1}^m N_h(\mathbf{x}) H(\xi_1) \mathbf{a}_h^1 + \sum_{h=1}^m N_h(\mathbf{x}) H(\xi_2) \mathbf{a}_h^2 \\ & + \sum_{h=1}^m N_h(\mathbf{x}) H(\xi_1) H(\xi_2) \mathbf{a}_h^3\end{aligned}$$

Some researchers have proposed a more efficient approach that avoids the cross terms by introduction of modified signed distance functions (Zi *et al.* 2004).



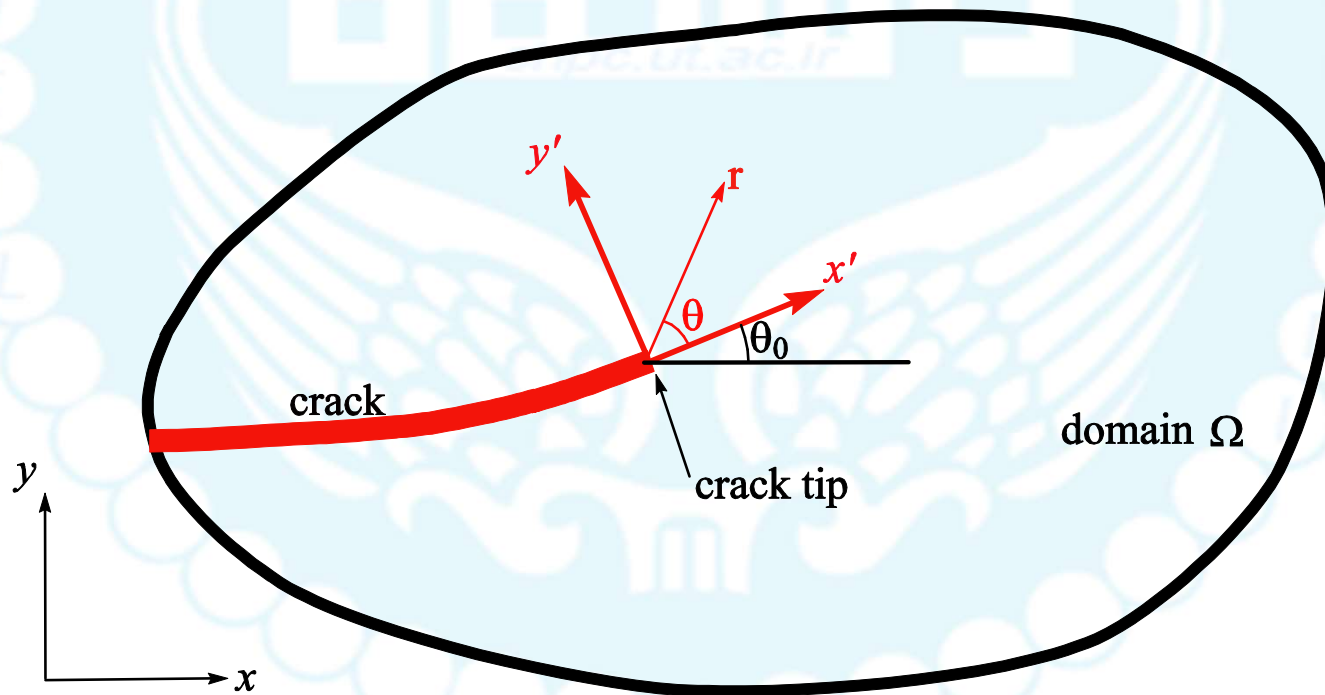
XFEM Crack Tip

Enrichments

Crack tip enrichments F_α

For the crack tip enrichments F_α in local crack tip polar coordinate system,

$$F_\alpha(r, \theta) = \left\{ \sqrt{r} \sin \frac{\theta}{2}, \sqrt{r} \cos \frac{\theta}{2}, \sqrt{r} \sin \theta \sin \frac{\theta}{2}, \sqrt{r} \sin \theta \cos \frac{\theta}{2} \right\}$$

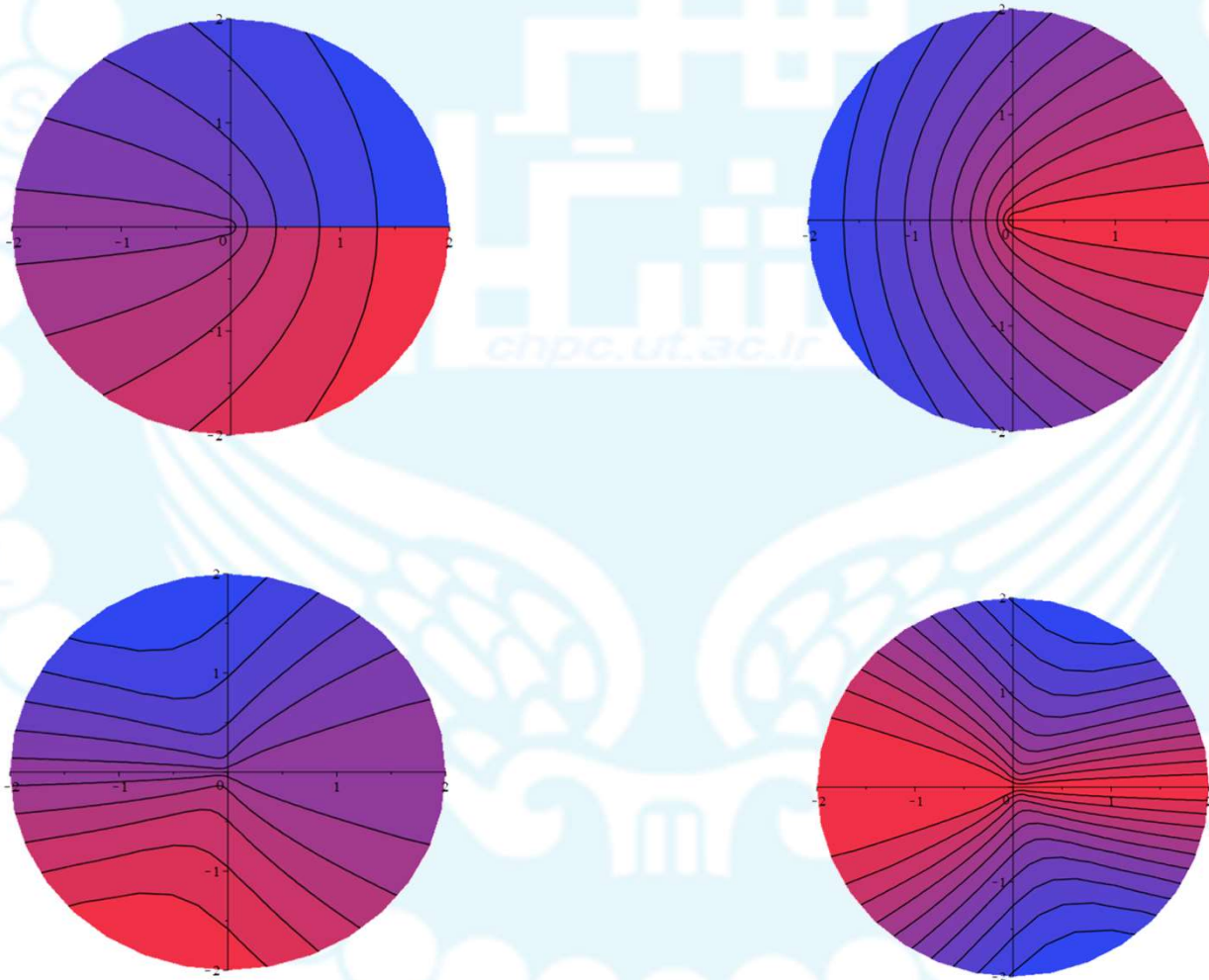


The first term of the crack-tip enrichment ensures the discontinuity across the crack edge. As a result, no strong discontinuity enrichment is simultaneously required for a crack-tip enriched element.

Homework: Draw the four crack tip enrichment functions over a cracked element (with the crack tip inside the element).

Several different sets of crack-tip enrichment functions are available for various problems, such as orthotropic materials, biomaterials, etc. They will be presented in a separate section.

The first enrichment function is discontinuous across the crack. As a result, no strong discontinuity enrichment is required in the crack-tip enriched element.





XFEM Enrichments

for

Weak Discontinuity

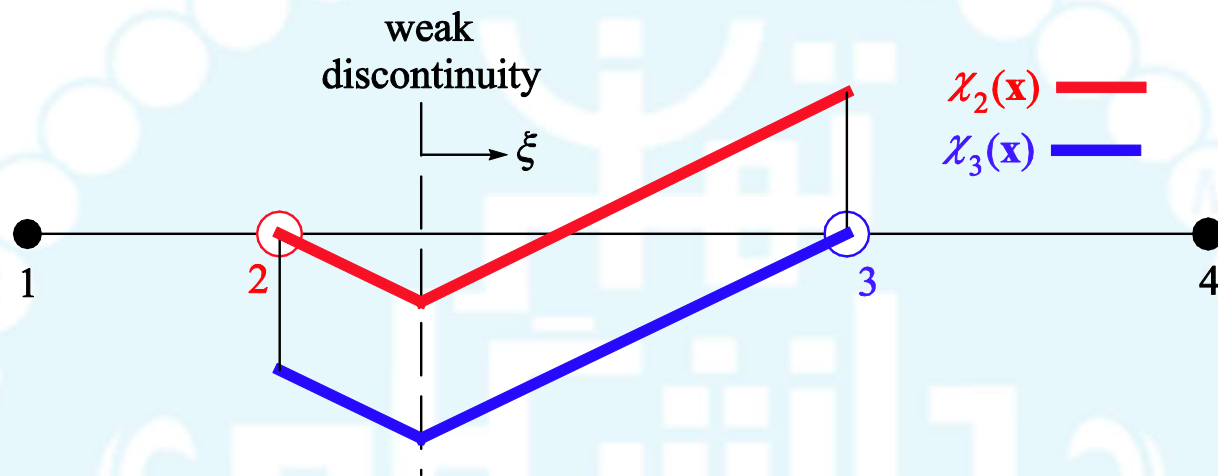
XFEM WEAK DISCONTINUITY ENRICHMENTS

The XFEM approximation can now be used for weak discontinuity problems by replacing the Heaviside function $H(\xi)$ with an appropriate enrichment function $\chi_k(\mathbf{x})$ (Bordas and Legay 2005)

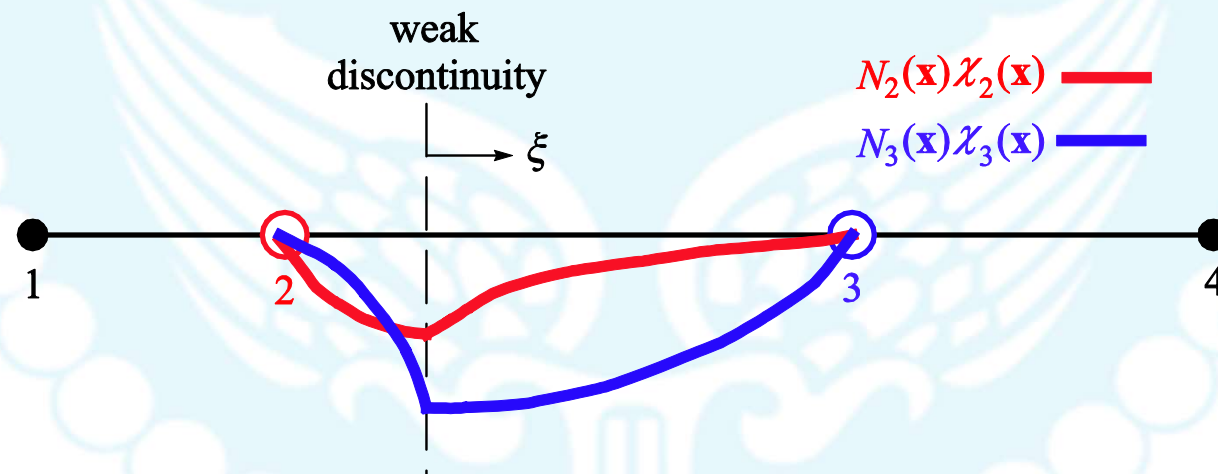
$$\mathbf{u}^h(\mathbf{x}) = \sum_{j=1}^n N_j(\mathbf{x}) \mathbf{u}_j + \sum_{k=1}^m N_k(\mathbf{x}) \chi_k(\mathbf{x}) \mathbf{a}_k$$

where $\chi_k(\mathbf{x})$ is the weak discontinuity enrichment function, associated with a node k of the finite element, defined in terms of the signed distance function $\xi(\mathbf{x})$:

$$\chi_k(\mathbf{x}) = |\xi(\mathbf{x})| - |\xi(\mathbf{x}_k)|$$

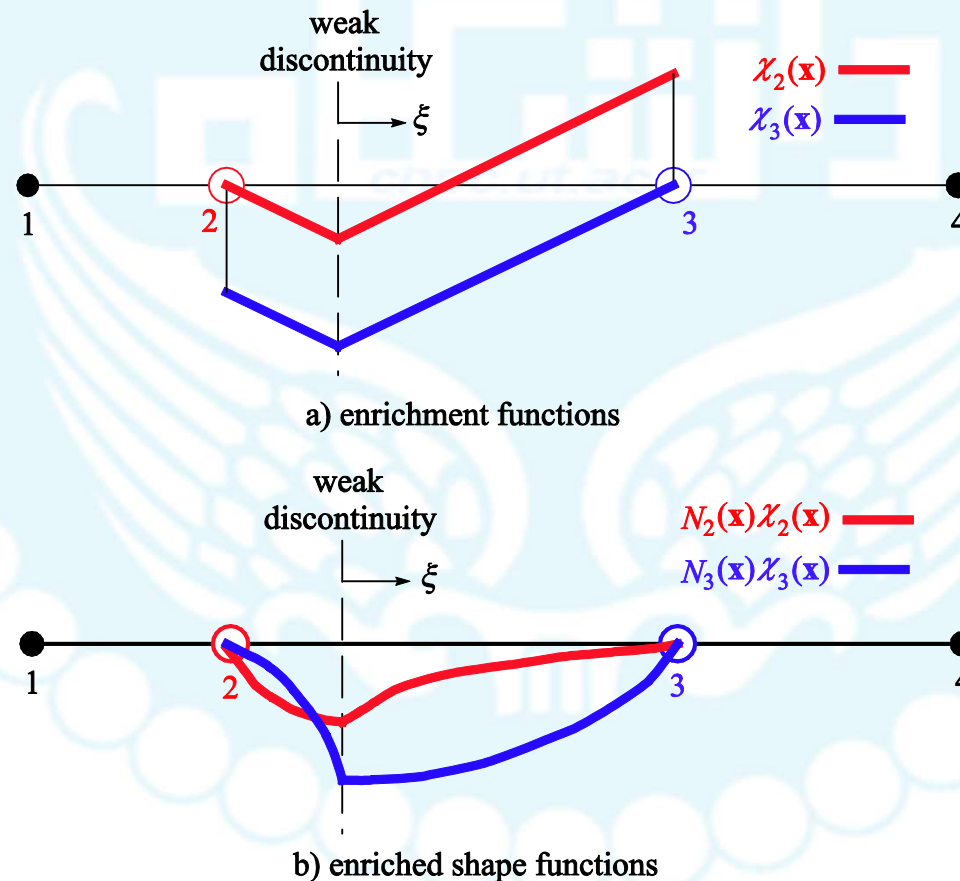


a) enrichment functions



b) enriched shape functions

This figure depicts how the original shape functions are transformed as an effect of enrichment by the weak discontinuous enrichment functions. Accordingly, a kink in the displacement field is introduced. As a result, a jump in its derivative, i.e. a discontinuity in the gradient of the function is anticipated.



The normal jump in the gradient field at the position of weak discontinuity can be computed as:

$$\left\langle \nabla \mathbf{u}^h(\mathbf{x}) \right\rangle = 2 \sum_{k=1}^m N_k(\mathbf{x}) \mathbf{a}_k$$

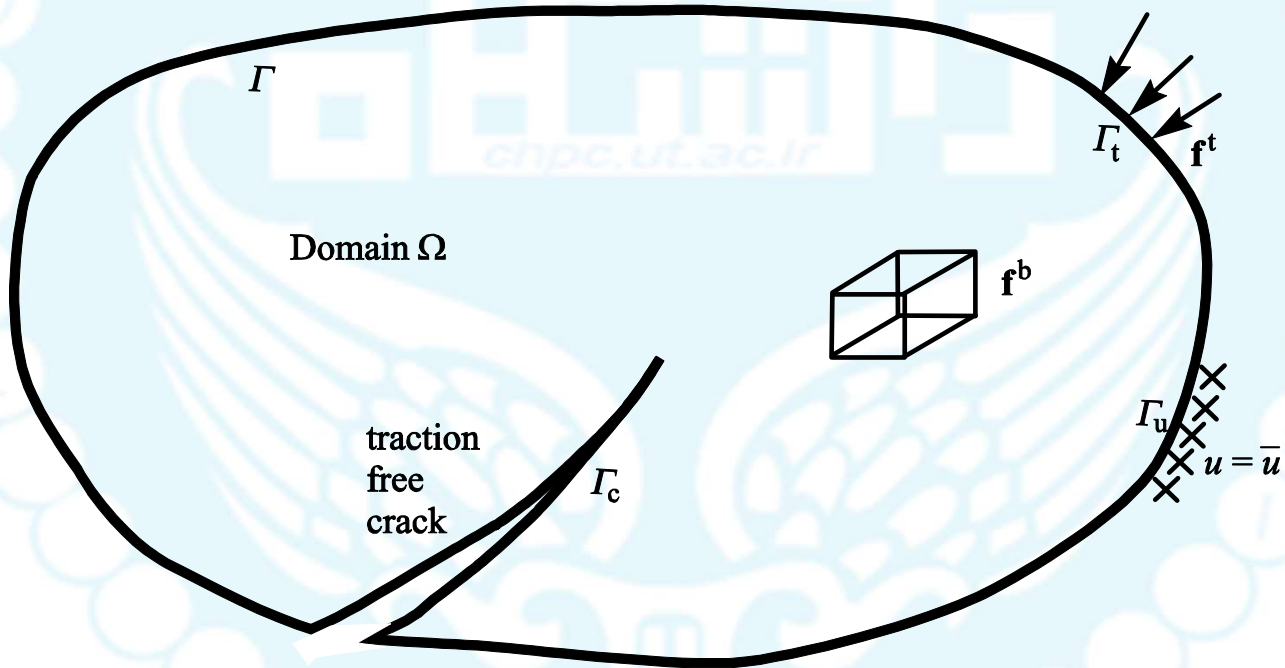
The weak discontinuity enriched shape functions vanish at both nodes 2 and 3. As a result, the interpolation is guaranteed without any shifting.

In addition, the direct effect of enrichment on displacements is limited only to the middle element which contains a weak discontinuity. However, an indirect influence exists due to different values of derivatives of enriched shape functions. A number of studies have directed towards new definitions of weak discontinuity enrichments to remove such indirect effects by ensuring vanishing derivatives of the enriched shape functions at nodal points (Esna Ashari and Mohammadi, 2011).

Discretized
Governing Equilibrium Equations

Governing equation

Consider a body in the state of equilibrium with the boundary conditions in the form of traction and displacement conditions.



The strong form of the equilibrium equation can be written as:

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{f}^b = 0 \quad \text{in } \Omega$$

with the following boundary conditions:

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{f}^t \quad \text{on } \Gamma_t \quad \text{external traction}$$

$$\mathbf{u} = \bar{\mathbf{u}} \quad \text{on } \Gamma_u \quad \text{prescribed displacement}$$

$$\boldsymbol{\sigma} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_c \quad \text{traction free crack}$$

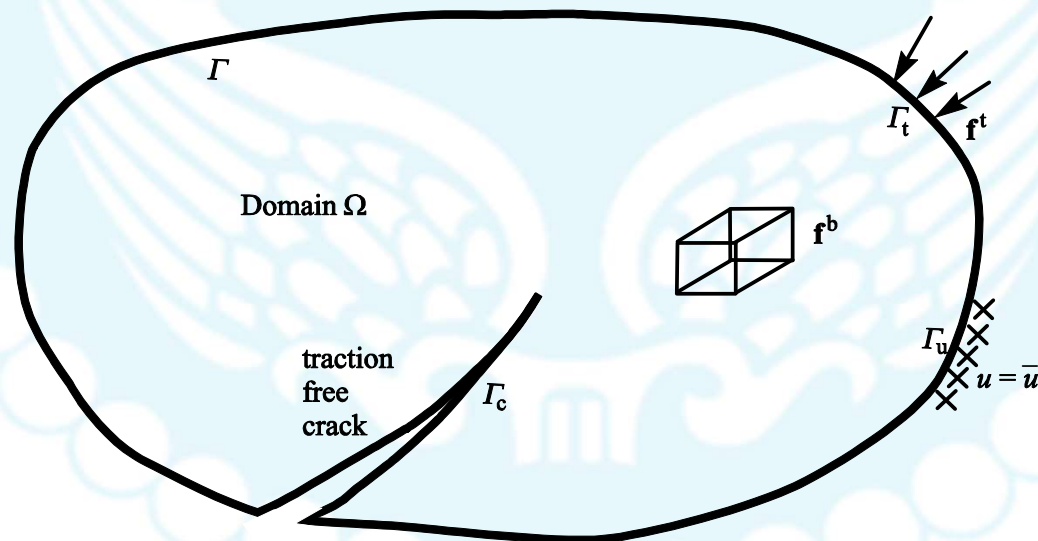
where Γ_t , Γ_u and Γ_c are traction, displacement and crack boundaries, respectively, $\boldsymbol{\sigma}$ is the stress tensor and $\mathbf{f}^b$ and $\mathbf{f}^t$ are the body force and external traction vectors, respectively.

The variational formulation of the boundary value problem can be defined as:

$$W^{\text{int}} = W^{\text{ext}}$$

or

$$\int_{\Omega} \boldsymbol{\sigma} \cdot \delta \boldsymbol{\varepsilon} \, d\Omega = \int_{\Omega} \mathbf{f}^b \cdot \delta \mathbf{u} \, d\Omega + \int_{\Gamma_t} \mathbf{f}^t \cdot \delta \mathbf{u} \, d\Gamma$$



XFEM discretisation

Discretisation of the weak form of the governing equation using the XFEM approximation results in a discrete system of linear equilibrium equations:

$$\mathbf{K}\mathbf{u}^h = \mathbf{f}$$

where $\mathbf{K}$ is the stiffness matrix, $\mathbf{u}^h$ is the vector of degrees of nodal freedom (for both classical and enriched ones) and $\mathbf{f}$ is the vector of external force.

The global matrix and vectors are calculated by assembling the matrix and vectors of each element. $\mathbf{K}$ and $\mathbf{f}$ for each element e are defined as

$$\mathbf{K}_{ij}^e = \begin{bmatrix} \mathbf{K}_{ij}^{uu} & \mathbf{K}_{ij}^{ua} & \mathbf{K}_{ij}^{ub} & \mathbf{K}_{ij}^{uc} \\ \mathbf{K}_{ij}^{au} & \mathbf{K}_{ij}^{aa} & \mathbf{K}_{ij}^{ab} & \mathbf{K}_{ij}^{ac} \\ \mathbf{K}_{ij}^{bu} & \mathbf{K}_{ij}^{ba} & \mathbf{K}_{ij}^{bb} & \mathbf{K}_{ij}^{bc} \\ \mathbf{K}_{ij}^{cu} & \mathbf{K}_{ij}^{ca} & \mathbf{K}_{ij}^{cb} & \mathbf{K}_{ij}^{cc} \end{bmatrix}$$

$$\mathbf{f}_i^e = \left\{ \mathbf{f}_i^u \quad \mathbf{f}_i^a \quad \mathbf{f}_i^{b1} \quad \mathbf{f}_i^{b2} \quad \mathbf{f}_i^{b3} \quad \mathbf{f}_i^{b4} \quad \mathbf{f}_i^c \right\}^T$$

and $\mathbf{u}^h$ is the vector of nodal parameters/DOFs:

$$\mathbf{u}^h = \left\{ \mathbf{u} \quad \mathbf{a} \quad \mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3 \quad \mathbf{b}_4 \quad \mathbf{c} \right\}^T$$

with

$$\mathbf{K}_{ij}^{rs} = \int_{\Omega^e} (\mathbf{B}_i^r)^T \mathbf{D} \mathbf{B}_j^s d\Omega \quad (r, s = \mathbf{u}, \mathbf{a}, \mathbf{b}, \mathbf{c})$$

$$\mathbf{K}_{ij}^{ab} = \mathbf{K}_{ij}^{ba} = \mathbf{K}_{ij}^{ac} = \mathbf{K}_{ij}^{ca} = 0 \quad (\text{usually})$$

and

$$\mathbf{f}_i^{\mathbf{u}} = \int_{\Gamma_t} N_i \mathbf{f}^t d\Gamma + \int_{\Omega^e} N_i \mathbf{f}^b d\Omega$$

$$\mathbf{f}_i^{\mathbf{a}} = \int_{\Gamma_t} N_i H \mathbf{f}^t d\Gamma + \int_{\Omega^e} N_i H \mathbf{f}^b d\Omega$$

$$\mathbf{f}_i^{\mathbf{b}^\alpha} = \int_{\Gamma_t} N_i F_\alpha \mathbf{f}^t d\Gamma + \int_{\Omega^e} N_i F_\alpha \mathbf{f}^b d\Omega \quad (\alpha = 1, 2, 3 \text{ and } 4)$$

$$\mathbf{f}_i^{\mathbf{c}} = \int_{\Gamma_t} N_i \chi \mathbf{f}^t d\Gamma + \int_{\Omega^e} N_i \chi \mathbf{f}^b d\Omega$$

B is the matrix of shape function derivatives,

$$\mathbf{B}_i^u = \begin{bmatrix} N_{i,x} & 0 \\ 0 & N_{i,y} \\ N_{i,y} & N_{i,x} \end{bmatrix}$$

$$\mathbf{B}_i^a = \begin{bmatrix} (N_i H(\xi))_{,x} & 0 \\ 0 & (N_i H(\xi))_{,y} \\ (N_i H(\xi))_{,y} & (N_i H(\xi))_{,x} \end{bmatrix}$$

$$\mathbf{B}_i^b = \begin{bmatrix} \mathbf{B}_i^{b1} & \mathbf{B}_i^{b2} & \mathbf{B}_i^{b3} & \mathbf{B}_i^{b4} \end{bmatrix} \quad \mathbf{B}_i^\alpha = \begin{bmatrix} (N_i F_\alpha)_{,x} & 0 \\ 0 & (N_i F_\alpha)_{,y} \\ (N_i F_\alpha)_{,y} & (N_i F_\alpha)_{,x} \end{bmatrix}, \quad (\alpha = 1, 2, 3 \text{ and } 4)$$

$$\mathbf{B}_i^c = \begin{bmatrix} (N_i \chi)_{,x} & 0 \\ 0 & (N_i \chi)_{,y} \\ (N_i \chi)_{,y} & (N_i \chi)_{,x} \end{bmatrix}$$

To include the effects of interpolation, the following shifting amendments are required:

$$\mathbf{B}_i^a = \begin{bmatrix} (N_i[H(\xi) - H(\xi_i)])_{,x} & 0 \\ 0 & (N_i[H(\xi) - H(\xi_i)])_{,y} \\ (N_i[H(\xi) - H(\xi_i)])_{,y} & (N_i[H(\xi) - H(\xi_i)])_{,x} \end{bmatrix}$$

$$\mathbf{B}_i^\alpha = \begin{bmatrix} [N_i(F_\alpha - F_{\alpha i})]_{,x} & 0 \\ 0 & [N_i(F_\alpha - F_{\alpha i})]_{,y} \\ [N_i(F_\alpha - F_{\alpha i})]_{,y} & [N_i(F_\alpha - F_{\alpha i})]_{,x} \end{bmatrix} \quad (\alpha = 1, 2, 3 \text{ and } 4)$$

and no shifting procedure is required for derivatives of weak enrichment functions.

In addition, the following vectors of nodal forces are modified to:

$$\mathbf{f}_i^{\mathbf{a}} = \int_{\Gamma_t} N_i [H(\xi) - H(\xi_i)] \mathbf{f}^t d\Gamma + \int_{\Omega^e} N_i [H(\xi) - H(\xi_i)] \mathbf{f}^b d\Omega$$

$$\mathbf{f}_i^{\mathbf{b}_\alpha} = \int_{\Gamma_t} N_i (F_\alpha - F_{\alpha i}) \mathbf{f}^t d\Gamma + \int_{\Omega^e} N_i (F_\alpha - F_{\alpha i}) \mathbf{f}^b d\Omega \quad (\alpha = 1, 2, 3 \text{ and } 4)$$

Derivatives of enrichment functions

Computation of the matrix of shape function derivatives **B** depends on the definition of the enrichment function.

Derivatives of strong discontinuity enrichment $\psi(x)=H(\xi)$

Derivative of the Heaviside function is the Dirac delta function:

$$H_{,i}(\xi) = \delta(\xi)$$

which vanishes except at the position of the crack interface:

$$H_{,i}(\xi) = \begin{cases} 1 & \text{at crack} \\ 0 & \text{otherwise} \end{cases}$$

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As a result,

$$\mathbf{B}_i^{\mathbf{a}} = \begin{bmatrix} N_{i,x}H(\xi) & 0 \\ 0 & N_{i,y}H(\xi) \\ N_{i,y}H(\xi) & N_{i,x}H(\xi) \end{bmatrix}$$

To include the effects of interpolation, $H(\xi)$ is replaced by $H(\xi) - H(\xi_i)$.

Derivatives of weak discontinuity enrichment $\psi(\mathbf{x}) = \chi(\mathbf{x}) = |\xi(\mathbf{x})|$

Derivative of the weak discontinuity enrichment function can be obtained from

$$\psi_{,i}(\mathbf{x}) = \text{sign}(\xi) \xi_{,i}(\mathbf{x})$$

Derivatives of $\xi(\mathbf{x})$ are calculated from the derivatives of the shape functions if a standard finite element interpolation is adopted to define the function in terms of its nodal values:

$$\xi(\mathbf{x}) = \sum_{j=1}^4 N_j(\mathbf{x}) \xi_j$$

$$\xi_{,i}(\mathbf{x}) = \sum_{j=1}^4 N_{j,i}(\mathbf{x}) \xi_j$$

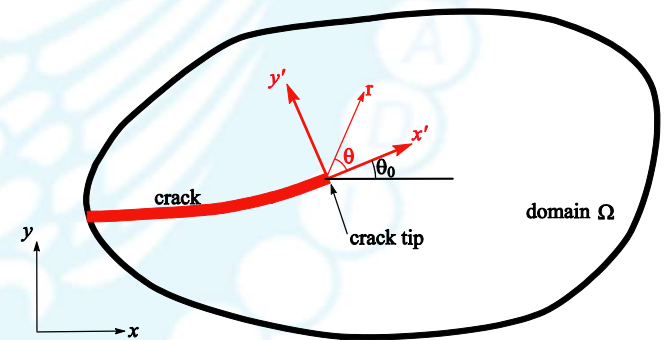
Derivatives of crack tip enrichment $\psi = F_\alpha(r, \theta)$

The near tip enrichment functions have already been defined in terms of the local crack tip polar coordinate system (r, θ) ,

$$F_\alpha(r, \theta) = \left\{ \sqrt{r} \sin \frac{\theta}{2}, \sqrt{r} \cos \frac{\theta}{2}, \sqrt{r} \sin \theta \sin \frac{\theta}{2}, \sqrt{r} \sin \theta \cos \frac{\theta}{2} \right\}$$

To compute the derivatives of crack tip enrichments in global coordinate systems:

- First, derivatives of $F_\alpha(r, \theta)$ with respect to the crack tip polar coordinates (r, θ) are computed.
- Then, they are transformed to local crack coordinates (x', y') . Finally, they are evaluated in the global (x, y) coordinates.



Derivatives of $F_{\alpha}(r,\theta)$ with respect to the crack tip polar coordinates (r,θ) become

$$F_{1,r} = \frac{1}{2\sqrt{r}} \sin \frac{\theta}{2}$$

$$F_{1,\theta} = \frac{\sqrt{r}}{2} \cos \frac{\theta}{2}$$

$$F_{2,r} = \frac{1}{2\sqrt{r}} \cos \frac{\theta}{2}$$

$$F_{2,\theta} = -\frac{\sqrt{r}}{2} \sin \frac{\theta}{2}$$

$$F_{3,r} = \frac{1}{2\sqrt{r}} \sin \frac{\theta}{2} \sin \theta$$

$$F_{3,\theta} = \sqrt{r} \left(\frac{1}{2} \cos \frac{\theta}{2} \sin \theta + \sin \frac{\theta}{2} \cos \theta \right)$$

$$F_{4,r} = \frac{1}{2\sqrt{r}} \cos \frac{\theta}{2} \sin \theta$$

$$F_{4,\theta} = \sqrt{r} \left(-\frac{1}{2} \sin \frac{\theta}{2} \sin \theta + \cos \frac{\theta}{2} \cos \theta \right)$$

Derivatives of $F_\alpha(r, \theta)$ with respect to the local crack coordinate system (x', y') can then be defined as

$$F_{1,x'} = -\frac{1}{2\sqrt{r}} \sin \frac{\theta}{2}$$

$$F_{1,y'} = \frac{1}{2\sqrt{r}} \cos \frac{\theta}{2}$$

$$F_{2,x'} = \frac{1}{2\sqrt{r}} \cos \frac{\theta}{2}$$

$$F_{2,y'} = \frac{1}{2\sqrt{r}} \sin \frac{\theta}{2}$$

$$F_{3,x'} = -\frac{1}{2\sqrt{r}} \sin \frac{3\theta}{2} \sin \theta$$

$$F_{3,y'} = \frac{1}{2\sqrt{r}} \left(\sin \frac{\theta}{2} + \sin \frac{3\theta}{2} \cos \theta \right)$$

$$F_{4,x'} = -\frac{1}{2\sqrt{r}} \cos \frac{3\theta}{2} \sin \theta$$

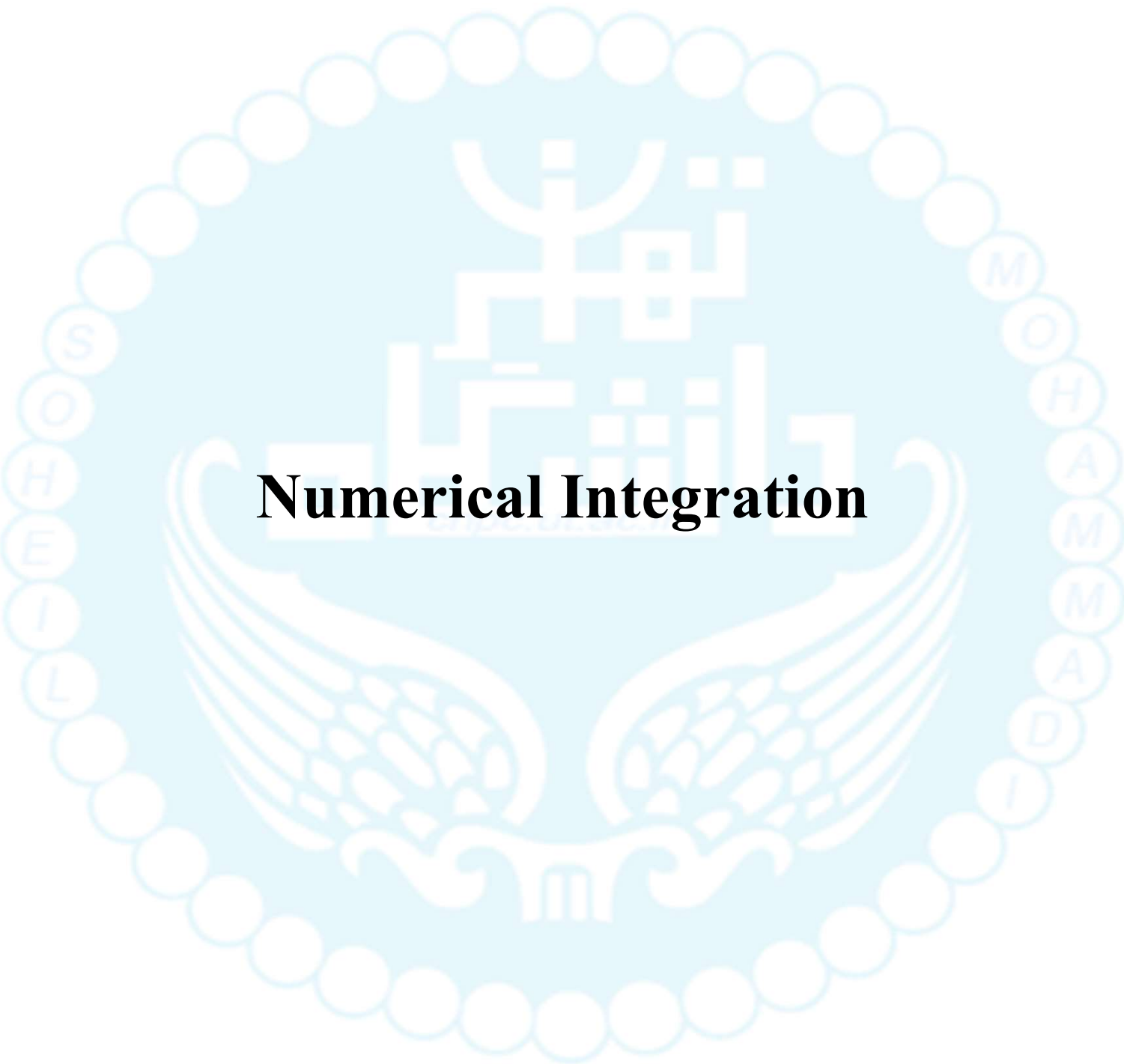
$$F_{4,y'} = \frac{1}{2\sqrt{r}} \left(\cos \frac{\theta}{2} + \cos \frac{3\theta}{2} \cos \theta \right)$$

Finally, the derivatives in the global coordinate system are obtained,

$$F_{\alpha,x} = F_{\alpha,x'} \cos(\theta_0) - F_{\alpha,y'} \sin(\theta_0)$$

$$F_{\alpha,y} = F_{\alpha,x'} \sin(\theta_0) + F_{\alpha,y'} \cos(\theta_0)$$

where θ_0 is the angle of crack path with respect to the x axis.

The logo of the Islamic University of Gaza is a circular emblem. It features a central shield with Arabic calligraphy. Above the shield is a crescent moon and a star. Below the shield are two large, stylized wings. The entire emblem is surrounded by a circular border containing the university's name in Arabic and English. The English text 'ISLAMIC UNIVERSITY OF GAZA' is written in a circular path around the emblem.

Numerical Integration

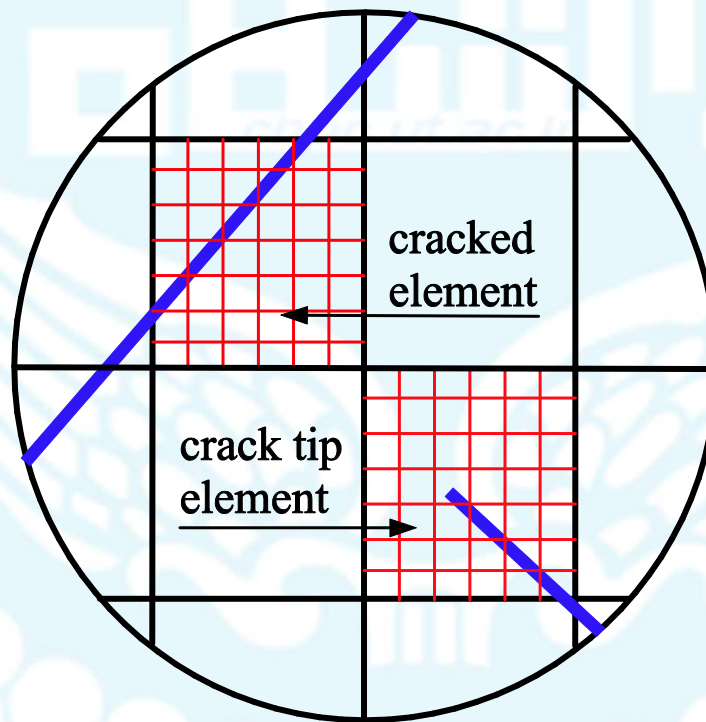
Numerical integration

The Gauss quadrature rule is widely used in finite element analysis for numerical evaluation of various integrals over a specified domain of interest such as a finite element. For polynomial integrands, the Gauss quadrature is proved to be exact. However, for non-polynomial integrands, it may result in substantial accuracy reduction.

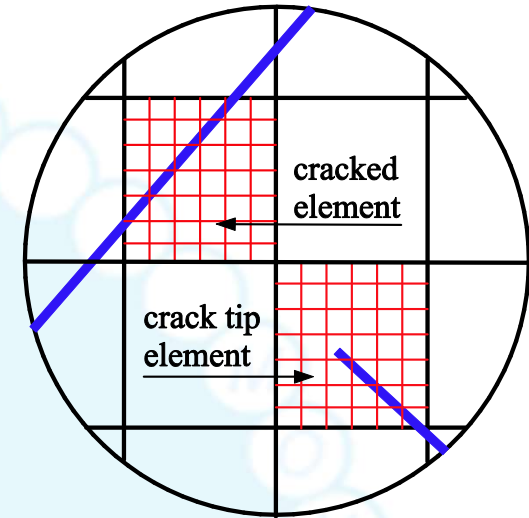
Introduction of singularity within a finite element transforms the displacement and stress fields into highly nonlinear fields. This is further complicated as the crack path turns to be substantially curved. On the other hand, existence of discontinuity within a finite element may similarly complicate the accurate integration procedure, even by conventional higher order quadrature rules. As a result, an efficient approach is required to define the necessary points needed for the integration within an enriched element. The approach has to be consistent with the geometry of the crack as well as the order of the enrichment functions.

Sub-quad technique

The first method, proposed by Dolbow (1999), is to subdivide the element into sub-quads. Each sub-quad is then integrated by a conventional Gauss integration rule. Different number of sub-quads can be used for elements that are cut by a crack or include a crack tip.



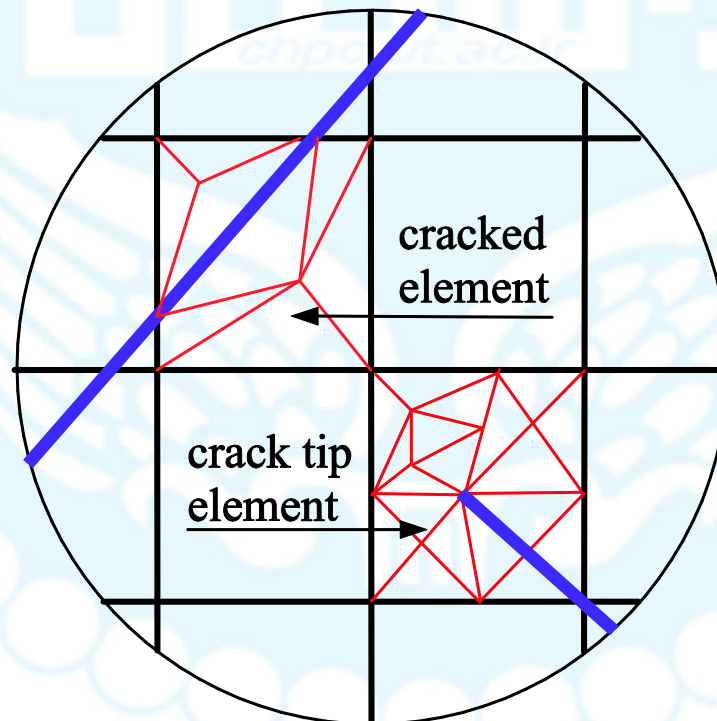
Sub-quad technique



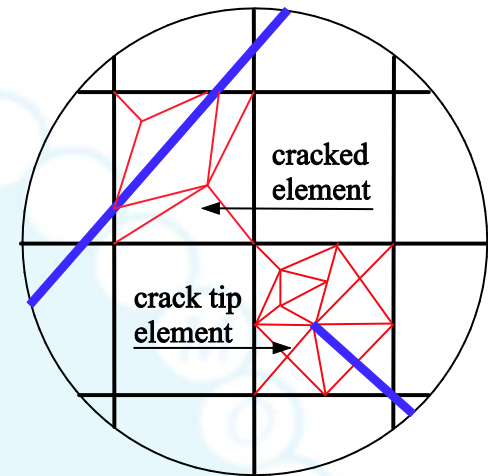
- While fewer gauss points are needed for elements that are cut by a crack, large number of Gauss points are required for crack tip elements.
- This method is not consistent with the fact that different sides of a cracked element do generate independent linear fields.
- The method is easy to implement.
- The increased computation is only a small fraction of total computational time.
- Some Gauss points, which may lay on the crack line, do not contribute to the stiffness matrix of the element.

Sub-triangle approach

The second approach, also proposed by Dolbow (1999), is to subdivide the element at both sides of the crack into sub-triangles whose edges are adapted to crack faces. For a crack tip element, more sub-triangles are required in front of the crack tip because of the existence of highly nonlinear and singular stress field. This method has been the most adopted integration approach in XFEM fracture analysis.



Sub-triangle approach

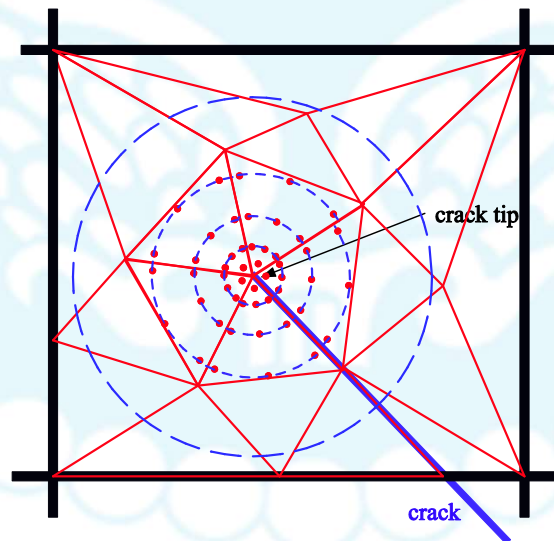


- Minimum triangles ensures accurate integration for an element cut by the crack. Moreover, independent displacement fields can well be generated across the crack.
- Fine triangulation is required for a crack tip element.
- Automatic triangulations routines are available in MATLAB and many other code developers.
- The increased computation is only a small fraction of total computational time.
- No Gauss points may be positioned on the crack line, avoiding numerical difficulties.

Polar distribution

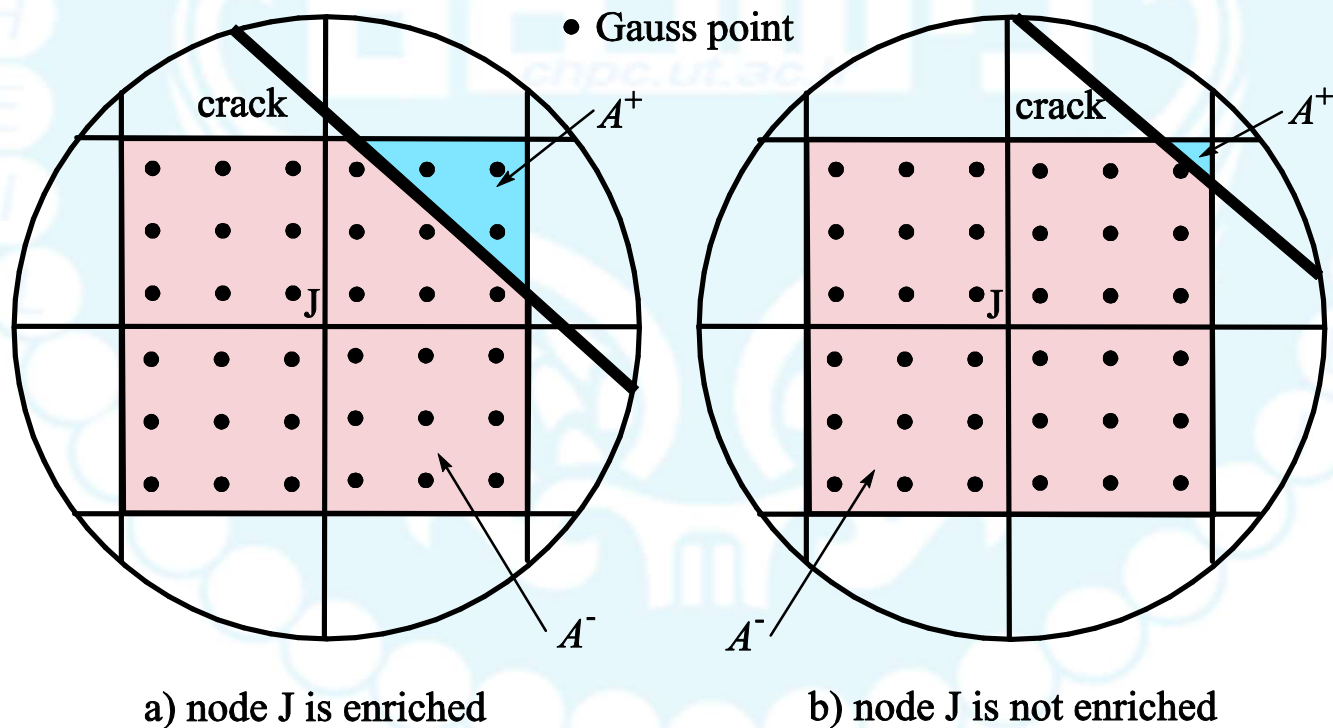
In recent years, a new idea has been developed to better fit the polar nature of stress distribution around the crack tip.

Accordingly, the distribution of Gauss points in a crack tip element is designed in a polar format, with closer points in the vicinity of the crack tip which gradually further distance each other towards the edges of the element. Such an approach has also been successfully used in crack analysis by the extended isogeometric analysis (XIGA) (Ghorashi *et al.*, 2012).



Limitations of enrichments

It is well-known that XFEM approximation may become ill-conditioned or even fail, if the enrichment is adopted for a case where the crack passes through or very close to a node. To avoid such a numerical difficulty, Dolbow (1999) proposed a simple geometric criterion to determine the candidacy of a node for enrichment.



In both methods, if values of $A^-/(A^++A^-)$ and $A^+/(A^++A^-)$, where A^+ and A^- are the area of the influence domain of a node above and below the crack, respectively, are smaller than an allowable tolerance value, the node must not be enriched.

A tolerance of 0.01% was proposed by Dolbow (1999), although it should be set according to each specific problem.

In an alternative approach, a node is enriched if each side of the crack in its influence domain includes at least one Gauss point.

